GUDLAVALLERU ENGINEERING COLLEGE

(An Autonomous Institute with Permanent Affiliation to JNTUK, Kakinada)

Seshadri Rao Knowledge Village, Gudlavalleru – 521 356.

Department of Electronics and Communication Engineering

HANDOUT

on

SIGNALS AND SYSTEMS

Vision

To be a leading centre of education and research in Electronics and Communication Engineering, making the students adaptable to changing technological and societal needs in a holistic learning environment.

Mission

- To produce knowledgeable and technologically competent engineers for providing services to the society and industry.
- To have a collaboration with leading academic, industrial and research organizations for promoting research activities among faculty and students.
- To create an integrated learning environment for sustained growth in electronics and communication engineering and related areas.

Program Educational Objectives

Graduates of the Electronics and Communication Engineering program will

1. demonstrate a progression in technical competence and leadership in the field of Electronics and Communication Engineering with professional ethics.
2. continue to learn and adapt to evolving technologies for catering to the needs of industry and society.

HANDOUT ON SIGNALS AND SYSTEMS

Class & Sem.: II B.Tech – I Semester Year : 2018-19

Branch : ECE Credits : 3

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1. **Brief History and Scope of the Subject**

   The knowledge of signals and systems has become an integral part of the electronics and communication curriculum. So it becomes very vital that a communication student study the subject Signals and systems very thoroughly. Apart from helping him in understanding his communication subjects well, it also serves as a good foundation course for more advance subjects like DSP& DIP.

   Signal processing being hailed as the most happening technology, the market has a high potential to absorb good professionals equipped with processing skills. If any student wants to pick up processing skills, a good starting point would be learning Signals and systems.

   The subject being a problematic one helps the student in honing his computational skills. So, finally, one can say that a good knowledge of signals eventually helps the student in becoming a good communication student.

2. **Pre-Requisites**
   - Mathematics - I
   - Mathematics - II

3. **Course Objectives:**
   - To familiarize with the basic concepts of signals and systems.
   - To introduce various transform techniques on signals.
   - To develop an understanding of signal transmission through LTI systems.

4. **Course Outcomes:**

   CO1: Classify the signals and various operations on signals.
   CO2: Perform Fourier analysis on the signals.
   CO3: Analyze the various systems.
   CO4: Perform correlation operation on signals.
   CO5: Apply the various sampling techniques on continues time signals.
   CO6: Analyze the various continues time signals through transformation techniques.

5. **Program Outcomes:**

Graduates of the Electronics and Communication Engineering Program will have

a. Apply knowledge of mathematics, science, and engineering for solving intricate engineering problems.
b. Identify, formulate and analyze complex engineering problems.
c. Design a system, component, or process to meet desired needs within realistic constraints such as economic, environmental, social, political, ethical, health and safety, manufacturability, and sustainability.
d. Design and conduct experiments, as well as to analyze and interpret data.
e. Use the techniques, skills, and modern engineering tools necessary for engineering practice.
f. Understand the impact of engineering solutions in a global, economic and societal context.
g. Design and develop eco-friendly systems, making optimal utilization of available natural resources.
h. Understand professional ethics and responsibilities.
i. work as a member and leader in a team in multidisciplinary environment
j. Communicate effectively.
k. manage the projects keeping in view the economical and societal considerations
l. Recognize the need for adapting to technological changes and engage in life-long learning.

6. **Mapping of Course Outcomes with Program Outcomes:**

	a	**b**	**c**	**d**	**e**	**f**	**G**	**h**	**i**	**j**	**k**
CO1	H				L						
CO2	H				L						
CO3	H				L						
CO4	H				L						
CO5	H				L						
CO6	H				L						

7. **Prescribed Text Books**
   a. Signals, Systems and Communications, B.P.Lathi
   b. Signals and Systems, A.V.Oppenheim, A.S.wilsky and S.H.Nawab

8. **Reference Text Books**
   a. Signals and Systems, Simon Haykin
   b. Fundamentals of Signals and System, Roberts.M.J

9. **URLs and Other E-Learning Resources**
   a. http://ieeexplore.ieee.org/Xplore/home.jsp
   b. Signals and Systems by Prof.S.C. Dutta Roy, IIT Delhi

10. **Digital Learning Materials:**
- IEEE Transactions on signal processing
- IEEE signal processing

11. Lecture Schedule / Lesson Plan

Topic	No. of Periods	
	Theory	Tutorial
UNIT –1: SIGNAL ANALYSIS		
Classification of Signals	3	
Basic Operations of Signals	3	2
Elementary Signals	2	
UNIT – 2:FOURIER SERIES REPRESENTATION OF CONTINUOUS TIME SIGNALS		
Relationship between Trigonometric and Exponential Fourier series	1	
Representation of a periodic function by the Fourier series over the entire interval	2	1
Convergence of Fourier series	1	
Alternate form of Trigonometric series	1	
Symmetry conditions	1	
Continuous time periodic signals	1	1
Properties of Fourier series	1	
Complex Fourier spectrum	1	
UNIT – 3: FOURIER TRANSFORMS		
Representation of an arbitrary function over the entire interval	1	
Fourier transform	1	1
Existences of Fourier transform	1	
Fourier transform of some useful functions	2	
Fourier transform of periodic function	1	
Properties of Fourier transform	1	1
Energy Density spectrum	1	
Parseval's Theorem	1	
SAMPLING:		
Sampling theorem for Band Limited Signals	1	
Reconstruction of signal from samples, Aliasing	1	1
Impulse sampling, Natural sampling, Flat top sampling	1	
UNIT – 4: LTI SYSTEMS		
Properties of systems	1	1
Linear Time Invariant (LTI) system	1	

Response of LTI system	1	
Convolution Integral, Graphical interpretation	1	
Properties of LTI system	1	
Transfer Function and frequency response of LTI system	1	
SIGNAL TRANSMISSION THROUGH LTI SYSTEMS		
Filter characteristics of LTI systems	1	2
Distortion less transmission through a system, signal bandwidth	1	
System bandwidth	1	
Ideal LPF, HPF and BPF characteristics	1	
Causality and physical realizability- Poly- Wiener criterion	1	
Relationship between bandwidth and rise time	1	
UNIT – 5: CORRELATION OF CONTINUOUS TIME SIGNALS		
Cross correlation and auto correlation of continuous time signals	2	2
Relation between convolution & correlation	1	
Properties of cross correlation and auto correlation	1	
Power spectral density	1	
Relation between correlation function and energy/power spectral density function	1	
UNIT – 6: LAPLACE TRANSFORMS		
Laplace transform of signals	1	1
Convergences of Laplace transform	1	
Properties of region of convergence (ROC)	1	
Unilateral Laplace transform	1	
Properties Unilateral Laplace transform	1	1
Inversion of Unilateral and Bilateral Laplace transform	1	
Relation between Laplace and Fourier Transforms	1	
Total No. of Periods:	**54**	**14**

12. Seminar Topics

- Examples of System

UNIT – I

Objective:

- To introduce the basic concepts of signals.

Syllabus: Classification of Signals, Basic Operations on Signals, Elementary Signals.

Outcomes:

Students will be able to

- Classify the signals.
- Perform basic operations on signals.

SIGNAL ANALYSIS

Signal

- Variation of one quantity with respect to one or more quantities.
- Function of one or more independent variables.
- Conveys some information.
- Examples –
    a) Voltage Signal, Current Signal, Speech signal, Biological signal.
    b) Variation of temperature with respect to time.
    c) Variation of share prices with respect to time.
    d) Variation of gold/ silver rate with respect to time.

Classification of Signals

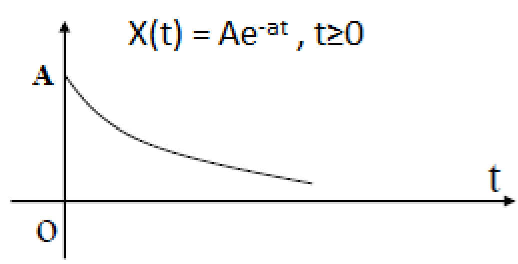
- Continuous and Discrete Time Signals.
- Even and Odd Signals.
- Periodic and Aperiodic Signals.
- Energy and Power Signals.
- Deterministic and Non Deterministic Signals.
- Causal and Non causal Signals.

Continuous and Discrete Time Signals

Continuous Time signal/Analog Signal

- Signal is defined at every instant of time

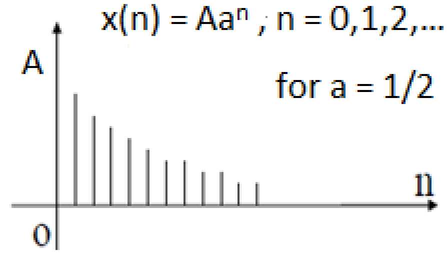
  Example: $x(t) = Ae^{-at}$, $t \geq 0$



Discrete Time Signal

- Signal is defined at discrete instants of time

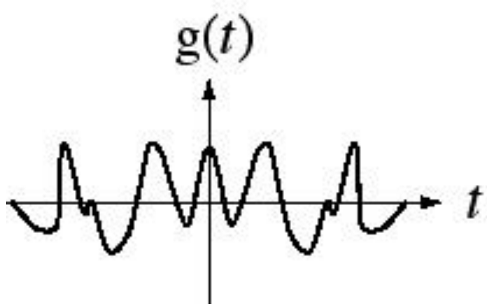
  Example



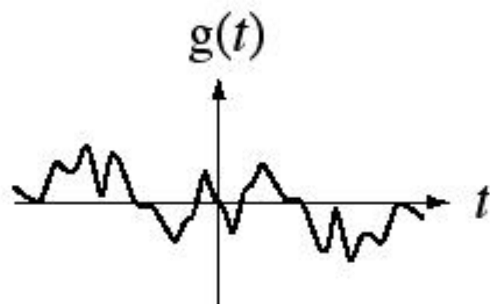
Even and Odd Signals

Even Signals: g(t) = g(-t)

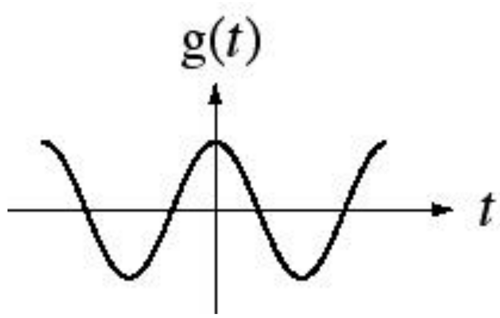
Odd Signals: g(t) = - g(-t)



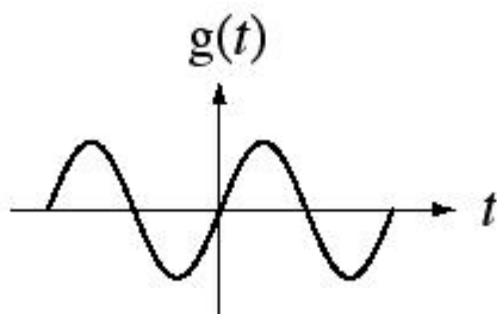
Even Signal



Odd Signal



Even Signal



Odd Signal

The **even part** of a function is $g_e(t) = \dfrac{g(t)+g(-t)}{2}$

The **odd part** of a function is $g_o(t) = \dfrac{g(t)-g(-t)}{2}$

A function whose even part is zero is odd.

A function whose odd part is zero is even.

The **even symmetry** is symmetrical with respect to vertical axis.

The **odd symmetry** is anti-symmetrical with respect to vertical axis.

Periodic and Aperiodic signals:

Periodic signal:

- x (t) is periodic if $x(t) = x(t+T)$

  Then $x(t+T) = x(t+2T) = x(t+3T)\ldots\ldots x(t+nT)$

  Then T is called Fundamental Period of x(t)
- Example

$$x(t) = A\cos(\omega t)$$

$$x(t+T) = A\cos[\omega(t+T)] = A\cos(\omega t+\omega T) = A\cos(\omega t+2\pi) = A\cos(\omega t)$$

$$\text{where } T = \frac{2\pi}{\omega} = \frac{1}{f}$$

Aperiodic/Non-periodic signal:

- For non-periodic signals

$x(t) \neq x(t+T)$

- A non-periodic signal is assumed to have period $T = \infty$
- Example of non-periodic signal is an exponential signal.

Energy and Power Signals

Energy Signal:

- A signal with finite energy and zero power is called Energy Signal i.e.for energy signal

$0 < E < \infty$ and $P = 0$

- Energy of a signal is defined as the *area under the square of the magnitude of the signal.*
- Energy of the signal can be calculated using formula

$$E = \int_{-\infty}^{\infty} |x(t)|^2\, dt$$

Power Signal:

- Some signals have infinite signal energy. In that case it is more convenient to deal with average signal power.
- For power signals

  $0 < P < \infty$ and $E = \infty$

- Average power of the signal x(t) is given by

$$P_x = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, dt$$

- For a periodic signal x(*t*) the average signal power is

$$P_x = \frac{1}{T} \int_T |x(t)|^2 \, dt$$

Where T is any period of the signal.

Periodic signals are generally power signals.

Generally infinite duration signals are power signals andfinite duration signals are energy signals.

A signal can be neither energy nor power signal.

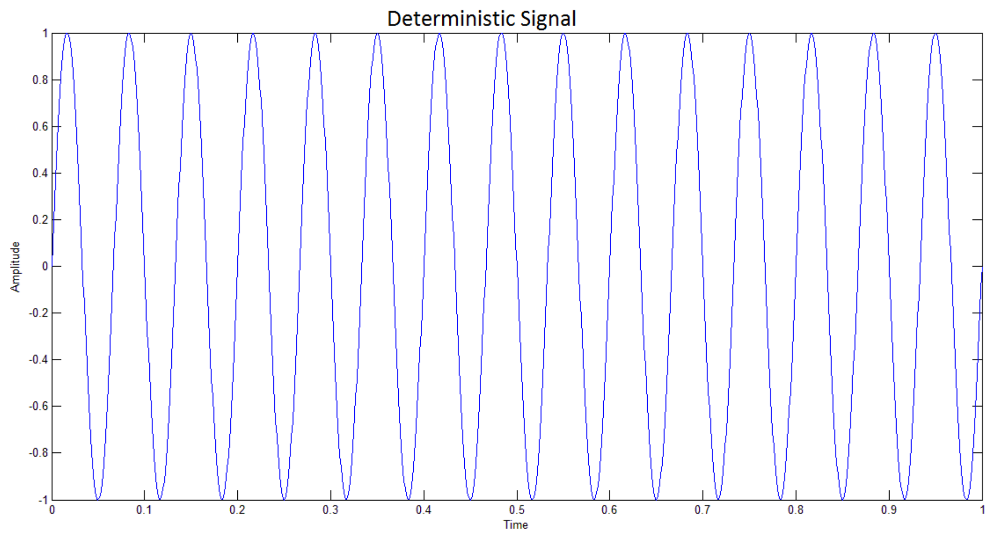
A signal cannot be both energy and power signal.

Deterministic &Non Deterministic Signals

Deterministic signals:

- Behavior of these signals can be predictable with respect to time.
- There is no uncertainty with respect to its value at any time.
- These signals can be expressed mathematically.

For example x(t) = sin(3t) is deterministic signal.



Non Deterministic or Random signals

- Behavior of these signals is random i.e. not predictable w.r.t time.
- There is an uncertainty with respect to its value at any time.
- These signals can't be expressed mathematically.
- For example Thermal Noise generated is non deterministic signal.

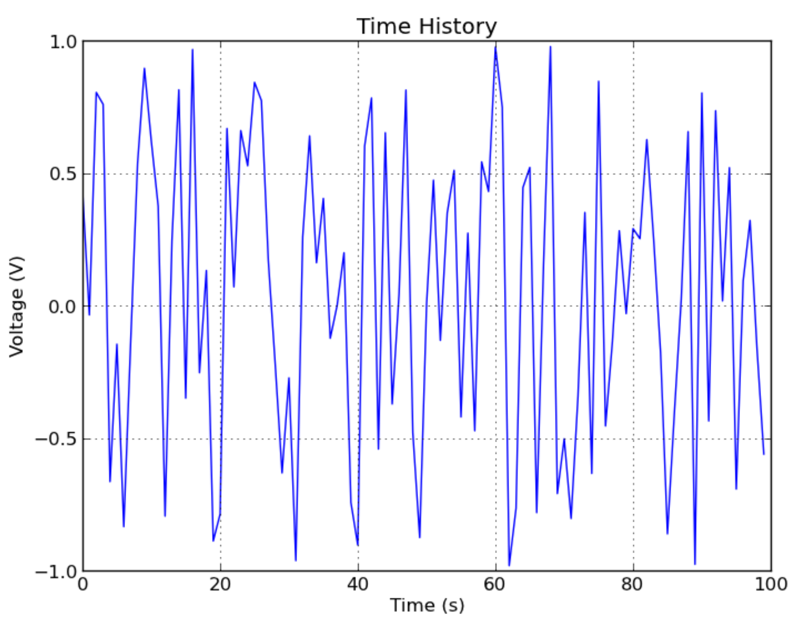


Figure: Random Signal

Causal and Non causal Signals

- Signal x(t) is said to be causal if x(t)=0 for t< 0.Otherwise non causal.
- Signal x(t) is said to be Anticausal if x(t)=0 for t> 0.

Basic Operations on Signals

If x(t) is a continuous signal then the following operations can be performed on dependent variable (amplitude) and independent variable (time) of a signal.

Operations on independent variable

1. **Time Scaling**

x(βt) is the time scaling version of x(t). where 'β' is a constant

if |β|<1 the signal will expand

if |β|>1 the signal will compressed

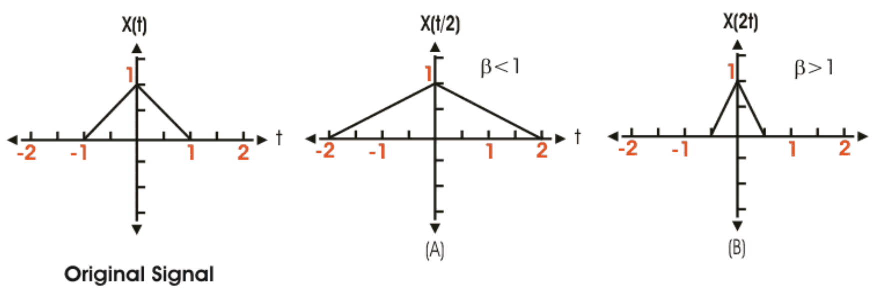
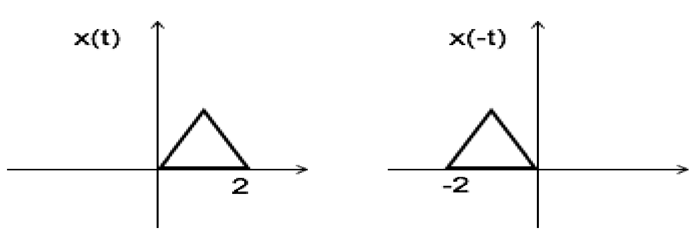


Figure (A): Expanded Signal
Figure (B): Compressed Signal

2. **Time Reversal**

Also called reflection of the signal. It is a special case of time scaling.

x(-t) is time reversal version of x(t)

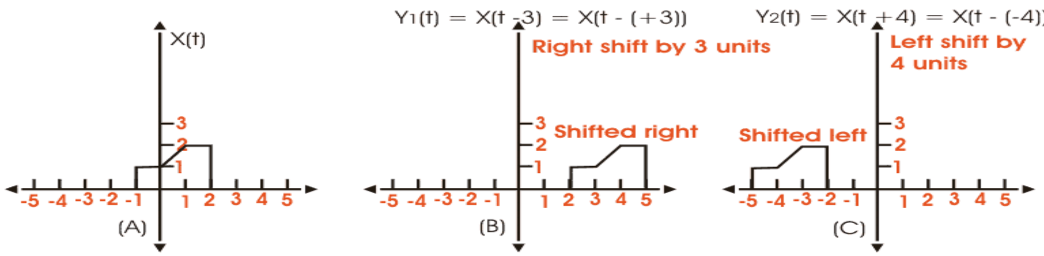


3. Time Shifting

$x(t - T)$ is the time shifting version of $x(t)$,where T is a constant.

If T is positive, the signal said to be delayed i.e., right shift operation.

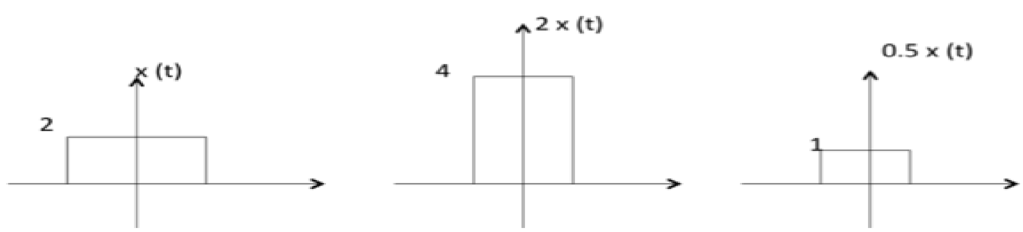
If T is negative, the signal is said to be advanced i.e., left shift operation.



Operations on dependent variable

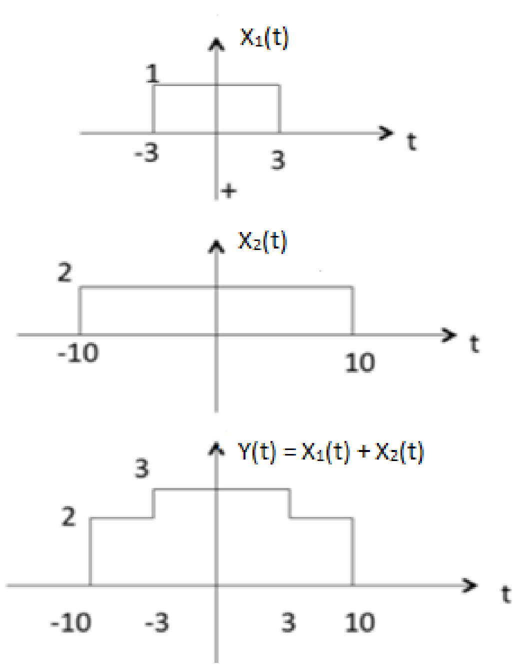
4. Amplitude Scaling

$Bx(t)$ is amplitude scaling of version of $x(t)$. Where 'B' is a constant, called scaling factor.

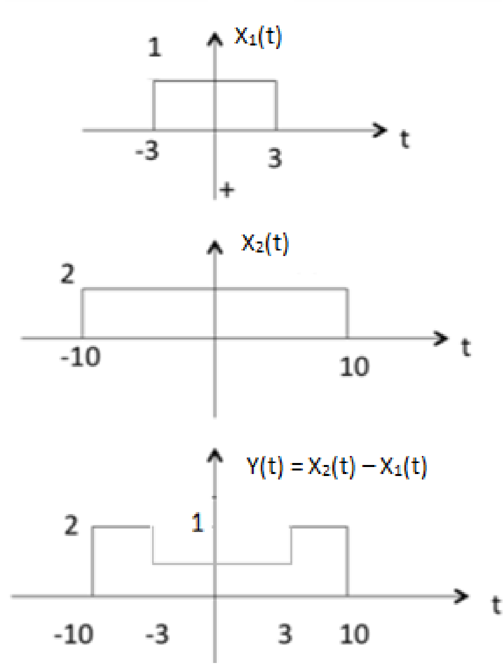


5. Addition/Subtraction:

$y(t) = x_1(t) \pm x_2(t)$ is called addition/subtraction operation

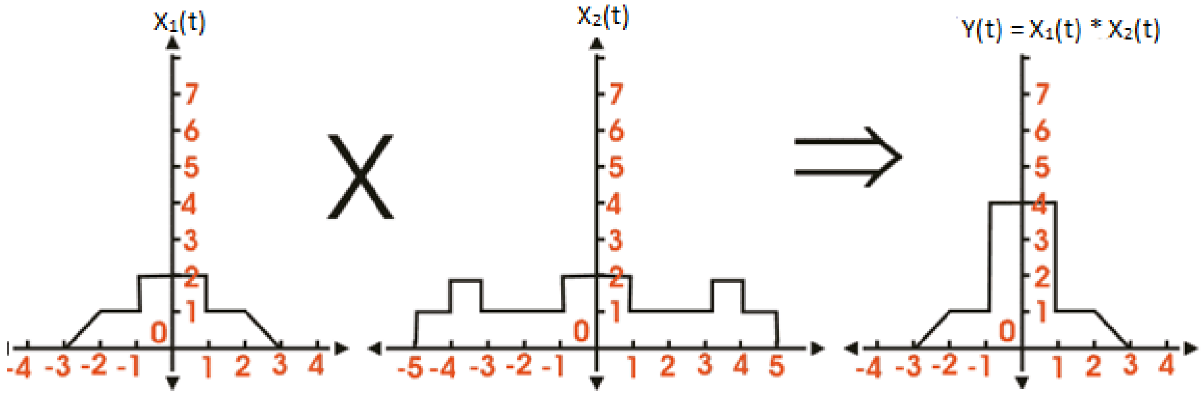


Addition operation



Subtraction operation

6. Multiplication: $Y(t) = X_1(t) * X_2(t)$ called multiplication operation



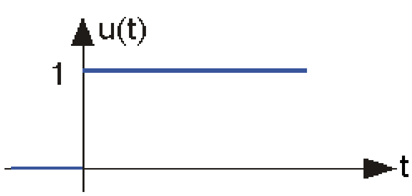
Elementary signals

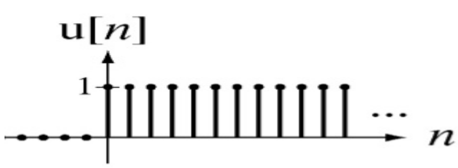
There are several elementary signals which play vital role in the study of signals and systems. These elementary signals serve as basic building blocks for the construction of more complex signals. These elementary signals are also called standard signals.

(a) **Unit Step signal**: It is denoted by u(t) and is defined as

$u(t) = 1 \quad \text{for} \quad t \geq 0$ $\qquad$ $u[n] = 1 \quad \text{for} \quad n = 0,1,2,\ldots.$

$= 0 \quad \text{for} \quad t < 0$ $\qquad$ $=0 \quad \text{for } n < 0$





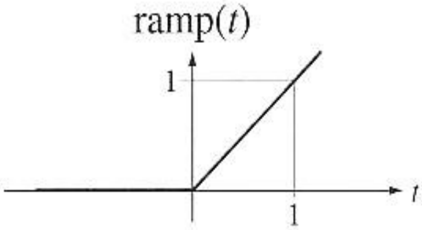
(b) **Unit ramp signal**: Denoted by r(t) or ramp(t) and defined as

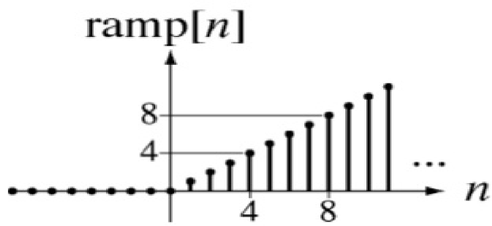
$r(t) = t \quad \text{for} \quad t \geq 0$ $\qquad$ $r[n] = n \quad \text{for} \quad n = 0,1,2,\ldots.$

$= 0 \quad \text{for} \quad t < 0$ $\qquad$ $= 0 \quad \text{for } n < 0$

Or

$r(t) = t\, u(t)$





The unit ramp function has unit slope. It is a signal whose amplitue varies linearly. It can be obtained by integrating the unit step function.

$$r(t) = \int u(t)dt$$

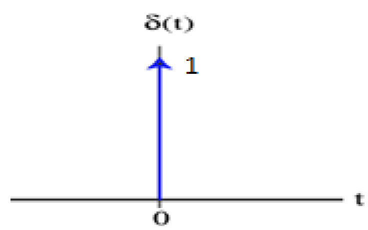
$$u(t) = \frac{d}{dt}r(t)$$

(c) **Unit impulse:** It plays very important role in communication system and analysis of systems. It is denoted by δ(t) . It is defined as

$\int_{-\infty}^{+\infty} \delta(t)dt \;= 1$ and

$\delta(t) = 0 \quad$ for $\quad t \neq 0$

The graphical representation for continuous impulse function is

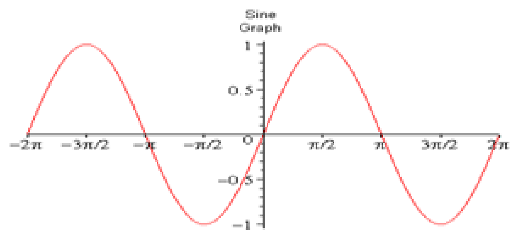


Properties of impulse function:

- It is even function of time t, i.e. $\delta(t) = \delta(-t)$
- $\int_{-\infty}^{+\infty} x(t)\,\delta(t)\,dt = x(0)$; $\int_{-\infty}^{\infty} x(t)\delta(t-t_0)dt = x(t_0)$
- $\delta(at) = \frac{1}{|a|}\delta(t)$
- $x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$

(d) **Sinusoidal signal**: it is defined as $x(t) = A\sin(\omega t + \theta)$ or $A\sin(\omega t)$
where A = amplitude
$\omega = 2\pi f$, angular frequency

θ = phase angle

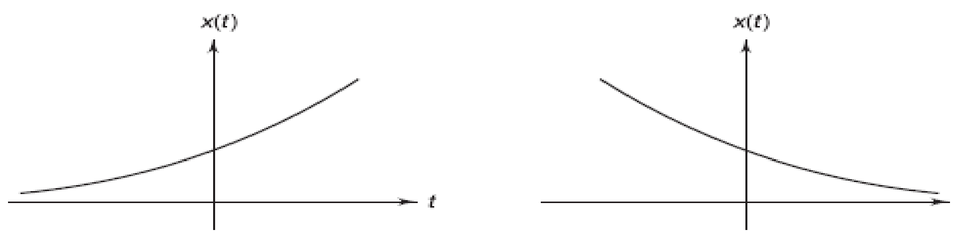


X(t) = Asin(ωt)

(e) **Exponential function**: An exponential function is defined as

x(t)= $e^{\sigma t}$ for all t ,Where 'σ' is constant

- If constant 'σ ' is real it is real exponential

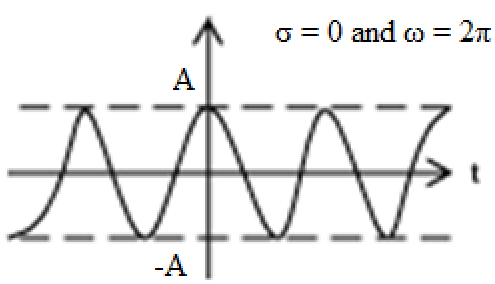


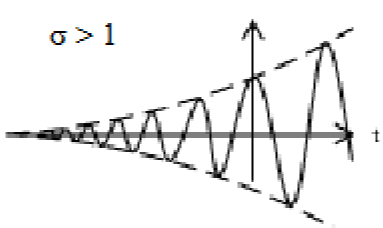
(a) Increasing ($\sigma > 0$)   (b) Decreasing ($\sigma < 0$)

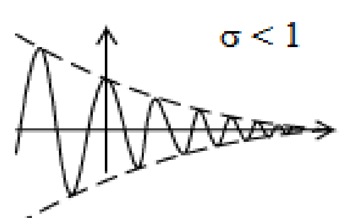
Real exponential function

- If constant 'σ ' is complex it is complex exponential

$x(t) = Ae^{st}$ where $s = \sigma + j\omega$







σ = 0, the real and imaginary parts of complex exponential signals are sinusoidal,

σ >1, the amplitude of the sinusoidal signal exponentially grows,

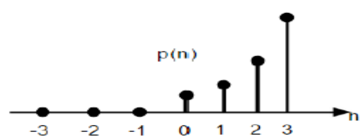
σ <1, the amplitude of the sinusoidal signals exponentially decays.

Other Signals

(a) **Unit parabolic signal**: Denoted by p(t) , defined as

$p(t) = t^2/2$ for $t \geq 0$ $\qquad$ $p(n) = n^2/2$ for $n=0,1,2,3,\ldots$

$= 0$ for $t \leq 0$ $\qquad$ $= 0$ for $n \leq 0$



The unit parabolic function can be obtained by integrating the unit ramp function or double integrating the unit step function.

$$p(t) = \iint u(t)dt = \int r(t)dt$$

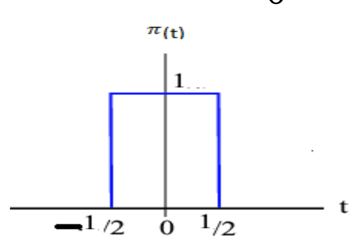
$$r(t) = \frac{d}{dt}p(t)$$

$$u(t) = \frac{d^2}{dt^2}p(t)$$

(b) **Rectangular pulse function**: it is denoted by $\pi(t)$ defined as

$$\pi(t) = 1 \qquad for|t| \leq \frac{1}{2}$$

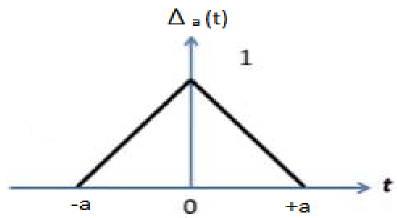
$$= 0 \qquad \text{otherwise}$$



It is an even function of t.

(c) **Triangular Pulse function**: it is denoted by $\Delta_a(t)$ and defined as

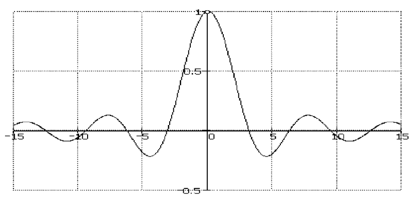
$$\Delta_a(t) = 1 - \frac{|t|}{a} \quad , \ |t| < a$$
$$= 0 \quad , \ |t| > a$$



It is an even function of t.

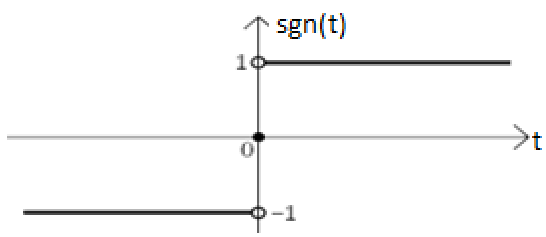
(d) **Sinc function**: Sinc (t) defined as

$$\text{Sinc}(t) = \frac{\sin \pi t}{\pi t} \qquad -\infty < t < +\infty$$



(e) **Signum function** : it is denoted by sgn(t), defind as

sgn(t) = +1 for t > 0
= -1 for t < 0



The signum function can be expressed in terms of unit step function as

sgn(t) = -1+ 2 u(t)

Assignment-Cum-Tutorial Questions

Section-A

1) Match the following [ ]

Signal	Notation
(i) Unit Step	(A) sgn(t)
(ii) Unit Impulse	(B) r(t)
(iii) Unit ramp signal	(C) δ(t)
(iv) Signum function	(D) u(t)

(a) i-A ii-B iii-C iv-D

(b) i-B ii-D iii-A iv-C

(c) i-D ii-C iii-B iv-A

(d) i-D ii-B iii-C iv-A

2) Match the following [ ]

Signal	Notation
(i) Unit Step	(A) -1+ 2 u(t)
(ii) Rectangular pulse	(B) $\int u(t)dt$
(iii) Unit ramp signal	(C) π(t)
(iv) Signum function	(D) $\frac{d^2}{dt^2} p(t)$

(a) i-A ii-B iii-C iv-D

(b) i-B ii-D iii-A iv-C

(c) i-D ii-C iii-B iv-A

(d) i-D ii-B iii-C iv-A

3) Which of the below statements are correct? [ ]

X: Sinc function is a symmetric function.

Y: Sinc function is having maximum value at t = 0

a) Only X
b) Only Y
c) Both X and Y
d) Neither X nor Y

4) Match the following [ ]

(i) Unit Step (A) differentiation of ramp

(ii) Sinc signal (B) even signal

(iii) Unit ramp signal (C) integration of unit step signal

(iv) Signum function (D) odd signal

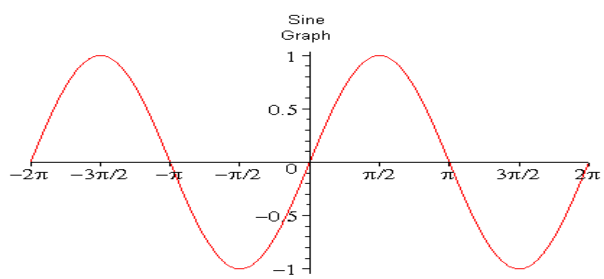
(a) i-A ii-B iii-C iv-D

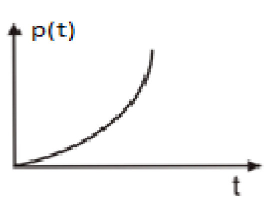
(b) i-B ii-D iii-A iv-C

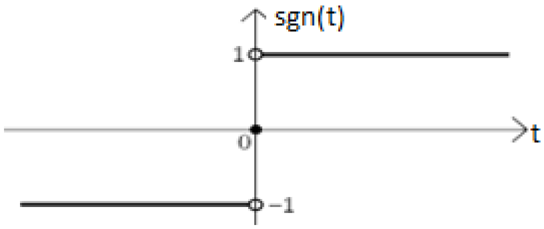
(c) i-D ii-C iii-B iv-A

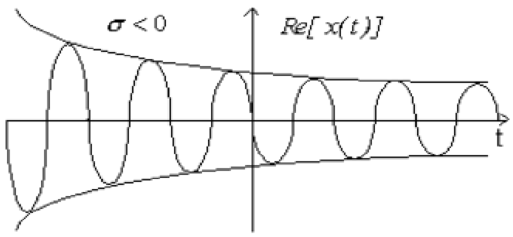
(d) i-D ii-B iii-C iv-A

5) Identify the signum function given below [ ]

a) 

b) 

c) 

d) 

6) Which of the below statements are correct? [ ]

X: Unit step signal is causal signal.

Y: Signum function is non-causal signal.

a) Only X
b) Only Y
c) Both X and Y
d) Neither X nor Y

7) Match the following [ ]

(i) Deterministic signal (A) Thermal noise

(ii) Random signal (B) r(t)

(iii) Causal signal (C) sin wt

(iv) Non-causal signal (D) u(-t)

(a) i-A ii-B iii-C iv-D

(b) i-C ii-A iii-B iv-D

(c) i-D ii-C iii-B iv-A

(d) i-D ii-B iii-C iv-A

8) For energy signal, energy and power are respectively as follows [ ]

a) Energy is finite and Power is finite
b) Energy is infinite and Power is infinite
c) Energy is zero and Power is finite
d) Energy is finite and Power is zero

9) Which of the below statements are correct? [ ]

X: All periodic signals are power signals.

Y: All aperiodic signals are energy signals.

a) Only X
b) Only Y
c) Both X and Y
d) Neither X nor Y

10) Match the following [ ]

(i) Time Shifting	(A) A x(t)
(ii) Amplitude Scaling	(B) x(t+T)
(iii) Time Reversal	(C) x(2t)
(iv) Time Scaling	(D) x(-t)

(a) i-A ii-B iii-C iv-D

(b) i-B ii-A iii-D iv-C

(c) i-D ii-C iii-B iv-A

(d) i-D ii-B iii-C iv-A

11) Which of the below statements are correct? [ ]

X: Time shifting operation on the signal advances or delayed the original signal.

Y: Time scaling operation the signal reverses the original signal.

a) Only X
b) Only Y
c) Both X and Y
d) Neither X nor Y

12) Match the following [ ]

(i) x(5t)	(A) Advance the signal
(ii) x(t+4)	(B) Delay the signal
(iii) x(0.5t)	(C) Compress the signal
(iv) x(t-7)	(D) Expand the signal

(a) i-A ii-B iii-C iv-D

(b) i-B ii-D iii-A iv-C

(c) i-C ii-A iii-D iv-B

(d) i-D ii-B iii-C iv-A

13) Which of the below statements are correct? [ ]

X: Unit impulse function is defined at t = 0.

Y: Unit impulse function is odd function.

a) Only X
b) Only Y
c) Both X and Y
d) Neither X nor Y

14.The value of the integral $\int_{-\propto}^{+\propto} \delta(at-b) =$ [ ]

(a) $\frac{1}{a^2}$ (b) $\frac{1}{|a|}$(c) 1 (d) 0

15. Evaluate$\int_{-\infty}^{\infty} t^2$ δ(t-6) dt. [ ]

(a)36(b)34 (c)35 (d) 37

16.A signal x(t) = A cos(ω_0t+Ø) is [ ]

a) An energy signal
b) A power signal
c) An energy as well as a power signal
d) neither an energy nor a power signal

17.The time period of the signal x(t)= cos60πt+sin50πt is [ ]

a) 2sec
b) 5sec
c) 10sec
d) 0.2sec

18. The rms value of x(t) = 10 cos5t cos10t is [ ]

a) 5
b) 25
c) 10
d) 50

19.The rms value of x(t) = A e^{j5t} is [ ]

a) A
b) A2
c) 2A

d) √ A

20.The power of x(t) = 8 cos 4t cos 6t is [ ]

a) 16
b) 4
c) 24
d) 8

21.The signal $e^{3t}u(t-2)$ is a [ ]

a) Causal
b) Non-causal
c) both a & b
d) None

Section-B

1. Explain the following Signals

(i) Energy signal and Power signals (ii)Periodic and Aperiodic Signals (iii) Deterministic and Random Signals (iv)Even and Odd Signals (v)Causal and Non causal Signals (vi) Continuous and Discrete time Signals

2. Discuss various operations performed on a signal? Give examples

3. Explain the following elementary signals

(a) Unit step(b) Exponential Signal (c) Unit ramp (d) Unit Impulse (e) Unit parabolic (f) Sinusoidal Signal

4. Evaluate the following integrals.

a) $\int_{-\infty}^{\infty} e^{-at^2}\delta(t-5)dt$ b) $\int_{-\infty}^{\infty} t^2\delta(t-6)dt$

5. Determine the power and rms value of the signal x (t) = A sin (ω_0t+Ø).

6. Calculate the even and odd components of the following signals:

a) x(t) = e^{j2t}
b) x(t) = cos (ω_0t + π/3)

7. Examine whether the following signals are periodic or not? If periodic determine the fundamental period.

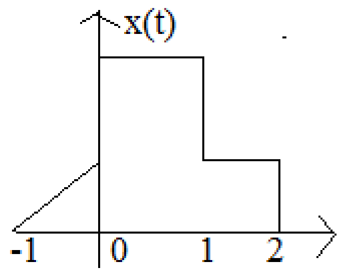
a) $e^{j4\pi t}$
b) b) Sin πt u(t)
c) c) cos 2t + sin √3t
d) d) 3 sin 200πt + 4 cos100πt

8. Draw the following signals

(i) u(-t+1) (ii) -2 r(t) (iii) Π(t+3)

9. Perform the following operations on the signal x(t) shown below

(i) x(t-2) (ii) x(2t+3) (iii) $x(\frac{3}{2}t)$



10. Determine whether the given signal is periodic or not and find the fundamental period.

$x(t) = (\cos 2\pi t)^2$

11. Find the energy of raised cosine pulse x(t) given as

$$x(t) = \frac{1}{2}[\cos(\omega t) + 1) \quad \frac{-\pi}{\omega} \leq t \leq \frac{\pi}{\omega}$$

$= 0$ otherwise

12. Determine the energy or power of the following signals

a. $x(t) = t \quad 0 \leq t \leq 1$

$= 2 - t \quad 1 \leq t \leq 2$

$= 0$ otherwise

b. $x(t) = 5\cos(\pi t) + \sin(5\pi t) \quad -\infty \leq t \leq \infty$

13. Find even and odd components of following signals

a) $x(t) = \cos\left[\Omega_0 t + \frac{\pi}{3}\right]$

b) $(1 + t^2 + t^3)\cos^2 10t$

14. Find neither whether the following signals are even or odd or neither even nor odd

a) x(t)=e^{-3t} b) u(t+4)-u(t-2)

Section-C

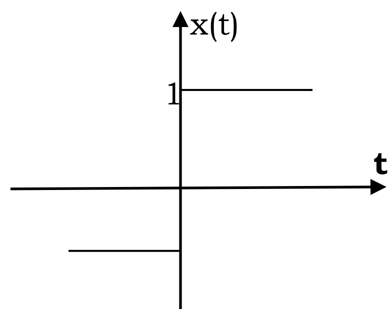
1.Let δ(t) denote the delta function. The value of the integral **GATE-10**

$\int_{-\infty}^{\infty} \cos(3t/2)\delta(t)\,dt$, is [ ]

(a) 1 (b) -1 (c) 0 (d) π/2

2. The function x(t) is shown in Fig. Even and odd parts of a unit-step function u(t)are respectively, [ ]

GATE-11



-1

(a) 1/2, 1/2x(t) (b) -1/2, 1/2x(t)

(c) 1/2, -1/2x(t) (d) -1/2, -1/2x(t)

3.The Dirac delta function δ(t) is defined as [ ]

(a) δ(t) = 1, t = 0**GATE-14**

= 0, otherwise

(b) δ(t) = ∞, t = 0

= 0, otherwise

(c) δ(t) = 1, t = 0

= 0, otherwise and

$\int_{-\infty}^{\infty} \delta(t)dt = 1$

(d) δ(t) = ∞, t = 0

= 0, otherwise and

$\int_{-\infty}^{\infty} \delta(t)dt = 1$

4.For a periodic signal v(t) = 30 sin100t+10 cos300t+6 sin(500t+π/4),the fundamental frequency in rad/s is [ ]

(a) 100 (b) 300(c) 500(d) 1500 **GATE-15**

UNIT II

Fourier series Representation of Continuous Time Signals

Learning Objectives:

- To introduce the concept of various Fourier series
- To introduce the concept of convergence of Fourier series.
- To introduce the concept of representation of a periodic function by Fourier series

Learning Outcomes:

Students will be able to

- Apply symmetric conditions
- Represent the periodic function by Fourier series
- Obtain the relation between trigonometric and exponential Fourier series

Syllabus:

Trigonometric and exponential Fourier series, relationship between trigonometric and exponential Fourier series, representation of a periodic function by the Fourier series over the entire interval, convergence of Fourier series, alternative form of trigonometric series, **symmetry conditions:** Even, Odd and Half-wave symmetry. **Properties of Fourier series:** linearity, time scaling, time shifting, time reversal, differentiation, integration, modulation, convolution and Parseval's theorem. Complex Fourier transforms.

Fourier Series

- It was developed by **Joseph Fourier.**
- The representation of signals over a certain time interval in terms of the linear composition of orthogonal functions is called **Fourier series**.
- It is also called **Harmonic analysis.**

Trigonometric Fourier Series

- The representation of signals over a certain time interval $(t_0,\ t_0 + T)$ in terms of the linear composition of trigonometric functions is called **Trigonometric Fourier series.**
- The Trigonometric Fourier Series representation of x (t) over interval (t_1, t_2)

$$x(t) = a_o + \sum_{n=1}^{\infty}[a_n cosn\omega_o t + b_n sinn\omega_o t] \quad (t_0 < t < t_0 + T) \qquad - (1)$$

Wherea_n, b_n are called constants and a_o is called dc component and is given by

$$a_o = \frac{1}{T}\int_{t_o}^{t_o+T} x(t)dt \qquad - \qquad (2)$$

$$a_n = \frac{2}{T}\int_{t_o}^{t_o+T} x(t)cosn\omega_o tdt \qquad - \qquad (3)$$

$$b_n = \frac{2}{T}\int_{t_o}^{t_o+T} x(t)sinn\omega_o tdt \qquad - \qquad (4)$$

Exponential Fourier Series

- The representation of signals over a certain time interval in terms of the linear composition of exponential functions is called **Exponential Fourier series.**
- The exponential Fourier series of signal x (t) over the interval (t_0, $t_0 + T$)

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_o t} \qquad (t_0 < t < t_0 + T) \qquad - \qquad (5)$$

OR

$$x(t) = c_0 + \sum_{n=-\infty}^{-1} c_n e^{jn\omega_o t} + \sum_{n=1}^{\infty} c_n e^{jn\omega_o t} \qquad - \qquad (6)$$

Where T = $\frac{2\pi}{\omega_0}$

Relationship between Trigonometric and Exponential Fourier Series

The complex exponential Fourier series is given by

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_o t} \qquad - \quad (7)$$

$$= c_0 + \sum_{n=-\infty}^{-1} c_n e^{jn\omega_o t} + \sum_{n=1}^{\infty} c_n e^{jn\omega_o t} \qquad - \quad (8)$$

$$= c_0 + \sum_{n=1}^{\infty}(c_{-n} e^{-jn\omega_o t} + c_n e^{jn\omega_o t}) \qquad - \quad (9)$$

$$= c_0 + \sum_{n=1}^{\infty} c_{-n}\,(cosn\omega_o t - jsinn\omega_o t) + c_n(cosn\omega_o t + jsinn\omega_o t)) \qquad - \quad (10)$$

$$x(t) = c_0 + \sum_{n=1}^{\infty}(c_{-n} + c_n)\ cosn\omega_o t + j(c_n - c_{-n})sinn\omega_o t) \qquad - \quad (11)$$

from (1) and (11) :

$$a_0 = c_0$$

$$a_n = c_n + c_{-n}$$

$$b_n = j(c_n - c_{-n})$$

or

$$c_0 = a_0$$

$$c_n = \frac{1}{2}(a_n - jb_n)$$

$$c_{-n} = \frac{1}{2}(a_n + jb_n)$$

Representation of a Periodic Function by the Fourier Series over the Entire Interval

- From Exponential Fourier Series

  $x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_o t} \qquad (t_0 < t < t_0 + T)$

  $= x(t_0 + T)$

  Therefore $x(t)$ is periodic with period T, i.e., $x(t) = x(t_0 + T)$

- From the above case if $x(t)$ is a periodic signal over a period $(t_0 < t < t_0 + T)$

  Then $x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_o t}$ is also periodic over the period $(-\infty, \infty)$

- **Trigonometric Fourier Series** representation of $x(t)$ over interval $(-\infty, \infty)$

  $$x(t) = a_o + \sum_{n=1}^{\infty}[a_n cosn\omega_o t + b_n sinn\omega_o t] \quad (-\infty < t < \infty)$$

  Where a_n, b_n are called constants and a_o is called dc component and is given by

  $$a_o = \frac{1}{T}\int_{t_o}^{t_o+T} x(t)dt$$

  $$a_n = \frac{2}{T}\int_{t_o}^{t_o+T} x(t)cosn\omega_o tdt$$

  $$b_n = \frac{2}{T}\int_{t_o}^{t_o+T} x(t)sinn\omega_o tdt$$

- **Exponential Fourier Series** representation of $x(t)$ over interval $(-\infty, \infty)$

  $$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_o t} \qquad (-\infty < t < \infty) \qquad - \qquad (5)$$

Dirichlet's Conditions (Convergence of Fourier Series)

For the Fourier series to exist for a periodic signal x(t) must satisfy certain conditions and they are

1. x(t) must be a single valued function
2. It should possess finite number of maxima & minima in one period.
3. It should have finite number of discontinuities in one period.
4. It should be absolutely integral over one period i,e $\int_0^T |x(t)|dt < \infty$

Alternative form of Fourier Series (Cosine Fourier series)

We know trigonometric Fourier series of x(t) is

$$x(t) = a_0 + \sum_{n=1}^{\infty} [a_n cosn\omega_0 t + b_n sinn\omega_0 t]$$

$$x(t) = a_0 + \sum_{n=1}^{\infty} \sqrt{a_n^2 + b_n^2} \left[\frac{a_n}{\sqrt{a_n^2 + b_n^2}} cosn\omega_0 t + \frac{b_n}{\sqrt{a_n^2 + b_n^2}} sinn\omega_0 t \right]$$

Let $a_0 = A_0, A_n = \sqrt{a_n^2 + b_n^2}$, $\theta_n = -\tan^{-1}\frac{b_n}{a_n}$ then $\frac{a_n}{\sqrt{a_n^2+b_n^2}} = cos\theta_n$ and $\frac{b_n}{\sqrt{a_n^2+b_n^2}} = -sin\theta_n$

in the above equation of x(t) we get

$$x(t) = A_0 + \sum_{n=1}^{\infty} A_n [cos\theta_n cosn\omega_o t - sin\theta_n sinn\omega_o t]$$

$$x(t) = A_0 + \sum_{n=1}^{\infty} A_n [\cos (n\omega_o t + \theta_n]$$

- A_0 is called the dc component.
- The term A_n represents the amplitude coefficients (or) harmonic amplitudes (or) spectral amplitudes of the Fourier series and θ_n represents the phase coefficients (or) phase angles of Fourier series.
- The cosine form is also called as "Harmonic Form" (or) "Polar Form" Fourier series.

Symmetric conditions or Wave symmetry

If the periodic signal x(t) has some type of symmetry, then some of the trigonometric Fourier series coefficients may become zero and calculation of the coefficients becomes simple.

A periodic function x(t) may possess the following four types of symmetry.

1. Even symmetry
2. Odd symmetry
3. Half wave symmetry
4. Quarter wave symmetry

Even symmetry or Mirror symmetry

x(t) is said to possess even symmetry if

$$x(-t) = x(t)$$

When even symmetry exists, the trigonometric Fourier series coefficients of x(t) are

$$a_o = \frac{2}{T}\int_0^{T/2} x(t)dt$$

$$a_n = \frac{4}{T}\int_0^{T/2} x(t)cosn\omega_o t\ dt$$

$$b_n = 0$$

Odd symmetry or Rotation symmetry

x(t) is said to possess odd symmetry if

$$x(-t) = -x(t)$$

When odd symmetry exists, the trigonometric Fourier series coefficients of x(t) are

$$a_o = 0$$

$$a_n = 0$$

$$b_n = \frac{4}{T}\int_0^{T/2} x(t)sinn\omega_o t\ dt$$

Half wave symmetry

A periodic signal x(t) which satisfies the condition

$x(t) = -x\left(t \pm \frac{T}{2}\right)$is said to possess half wave symmetry.

The Fourier series expansion of x(t) contains odd harmonics only

When **"n"** is odd

$$a_n = \frac{4}{T}\int_0^{T/2} x(t)cosn\omega_o t\ dt$$

$$b_n = \frac{4}{T}\int_0^{T/2} x(t)sinn\omega_o t\ dt$$

When "n" is even

$$a_n = 0$$

$$b_n = 0$$

Quarter wave symmetry

x(t) is said to possess quarter wave symmetry if

$x(t) = x(-t)$or$x(t) = -x(-t)$ and also $x(t) = -x(t \pm T/2)$

i.e. if function x(t) has either even symmetry or odd symmetry along with half wave symmetry then it is said to have quarter wave symmetry.

The following two cases are possible:

Case 1:

If $x(t) = -x(-t)$ and $x(t) = -x(t \pm T/2)$ then

$$a_0 = 0$$

$$a_n = 0$$

$$b_n = \frac{8}{T}\int_0^{T/4} x(t) sinn\omega_o t \, dt \quad \text{(When n is even)}$$

Case 2:

If $x(t) = x(-t)$ and $x(t) = -x(t \pm T/2)$ then

$$a_0 = 0$$

$$a_n = \frac{8}{T}\int_0^{T/4} x(t) cosn\omega_o t \, dt$$

$$b_n = 0 \text{(When n is odd)}$$

Properties of Fourier series

Let us suppose $x_1(t) and \; x_2(t)$ are two periodic signals with period 'T' and Fourier series coefficients $C_n \; and \; D_n$ respectively.

1. Linearity Property

If

$$FS\{x_1(t)\} = C_n \; and \; FS\{x_2(t)\} = D_n$$

then

$$FS\{A\, x_1(t) + Bx_2(t)\} = AC_n + B\, D_n$$

2. Time Reversal

If

$$Fs\{x(t)\} = C_n$$

then

$$Fs\{x(-t)\} = C_{-n}$$

3. Time Shifting

If

$$Fs\{x(t)\} = C_n$$

then

$$FS\ \{x(t-t_0) =\ e^{-jn\omega_0 t_0} C_n$$

4. Time Scaling

If

$$Fs\{x(t)\} = C_n$$

then

$$Fs\{x(at)\} = C_n\ with\ \omega_0 \to a\omega_0$$

5. Convolution Property

If

$$FS\{x_1(t)\} = C_n\ and\ FS\{x_2(t)\} = D_n$$

then

$$FS\{x_1(t) * x_2(t)\} = C_n D_n$$

6. Time Differentiation Property

If

$$Fs\{x(t)\} = C_n$$

then

$$FS\left\{\frac{dx(t)}{dt}\right\} = jn\omega_0 C_n$$

7. Time Integration Property

If

$$Fs\{x(t)\} = C_n$$

then

$$FS\left\{\int_{-\infty}^{t} x(\tau)d\tau\right\} = \frac{C_n}{jn\omega_0}\ (\text{if } C_0 = 0)$$

8. Modulation or Multiplication Property

If

$$FS\{x_1(t)\} = C_n\ and\ FS\{x_2(t)\} = D_n$$

then

$$FS\{x_1(t)x_2(t)\} = \sum_{l=-\infty}^{\infty} C_l D_{n-l}$$

9. Parseval's Theorem

If

$$FS\{x_1(t)\} = C_n \; and \; FS\{x_2(t)\} = D_n$$

then parseval's theorem states that

$$\frac{1}{T}\int_{t_0}^{t_0+T} x_1(t)x_2^*(t)dt = \sum_{n=-\infty}^{\infty} C_n D_n^*$$

10. Conjugation and Conjugate Symmetry Property

If

$$Fs\{x(t)\} = C_n$$

then

$Fs\{x^*(t)\} = C_{-n}^*$ (For complex x(t))-----------Conjugate property

and

$C_{-n} = C_n^*$ (For real x(t))-----------Conjugate symmetry property

11. Symmetry properties

The following are the symmetry properties

$$Re\{C_n\} = Re\{C_{-n}\}$$

$$Im\{C_n\} = Im\{C_{-n}\}$$

- If x(t) is real $C_n = C_{-n}^*$.
- If x(t) is even, then x(t) = x(-t). This implies that $C_n = C_{-n}$. That is Fourier series coefficients are also even.
- If x(t) is real and even C_n also real and even.
- If x(t) is odd, then x(t) = -x(-t). This implies that $C_n = -C_{-n}$. That is Fourier series coefficients are also odd.
- If x(t) is real and odd, C_n is purely imaginary and odd.

Assignment-Cum-Tutorial Questions

<u>Section-A</u>

1. The Fourier Series of an odd Periodic function does not contain [ ]
   a) odd harmonics
   b) even harmonics
   c) cosine terms
   d) sine terms
2. Which of the below statements are correct? [ ]

   X: A function is said to be even if x(t) = x(-t).

   Y: A function is said to be odd if x(t) = -x(t).

   a) Only X
   b) Only Y
   c) Both X and Y
   d) Neither X nor Y
3. Match the following [ ]

i. Periodic signal spectrum.	A. Antisymmetrical about vertical axis
ii. Aperiodic signal spectrum.	B. Symmetrical about vertical axis
iii. Exponential FS amplitude spectrum.	C. Discrete
iv. Exponential FS phase spectrum.	D. Continuous

   a) i-A ii-B iii-C iv-D
   b) i-B ii-D iii-A iv-C
   c) i-C ii-D iii-B iv-A
   d) i-D ii-B iii-C iv-A
4. The trigonometric Fourier series for an even function contains only [ ]
   a) the dc terms
   b) cosine terms

c) sine terms

d) odd harmonics terms

5. Which of the following Dirichlets's conditions are incorrect [ ]

W: Function x(t) must be a single valued function.

X: Function x(t) has infinite number of maxima & minima.

Y: Function x(t) has infinite number of discontinuities.

Z: Function x(t) is absolutely integral over one period i.e. $\int_0^T |x(t)|dt < \infty$

a) W and Z

b) X and Y

c) Y and Z

d) X and Z

6. [ ]

W: The sum of two continuous-time periodic signals $x_1(t)$ & $x_2(t)$ with periods T_1 and T_2 may or may not be periodic depending on the relation between T_1 and T_2.

X: The sum of two periodic signals is periodic only if the ratio of their respective periods T_1/T_2 is a rational number or the ratio of two integers.

Y: The fundamental period is the least common multiple (LCM) of T_1 and T_2.

Z: If the ratio T_1/T_2 is a rational number, then the signals $x_1(t)$ & $x_2(t)$ do not have a common period and x(t) cannot be periodic.

a) Only X and Y are Correct.

b) Only Y and Z are Correct.

c) W, X and Y are Correct.

d) W, X, Y and Z are Correct.

7. Match the following [ ]

i. Periodic signal spectrum.	A. Antisymmetrical about vertical axis
ii. Aperiodic signal spectrum.	B. Symmetrical about vertical axis
iii. Exponential FS amplitude spectrum.	C. Discrete
iv. Exponential FS phase spectrum.	D. Continuous

a) i-A ii-B iii-C iv-D

b) i-B ii-D iii-A iv-C

c) i-C ii-D iii-B iv-A

d) i-D ii-B iii-C iv-A

8. X: A trigonometric Fourier series has a one-sided spectrum [ ]

   Y: An exponential Fourier series has a two sided spectrum

   a) Only X is true

   b) Only Y is true

   c) Both X and Y are true

   d) Both X and Y are false

9. Match the following trigonometric FS coefficients and exponential FS coefficients [ ]

i. C_n	A. $j(C_n - C_{-n})$
ii. C_{-n}	B. $C_n + C_{-n}$
iii. a_n	C. $\frac{1}{2}(a_n + jb_n)$
iv. b_n	D. $\frac{1}{2}(a_n - jb_n)$

   a) i-A ii-B iii-C iv-D

   b) i-B ii-D iii-A iv-C

   c) i-C ii-D iii-B iv-A

   d) i-D ii-C iii-B iv-A

10. The exponential Fourier series representation of a signal x(t) over the interval t_0 to $(t_0 + T)$ is given by [ ]

   a) $\sum_{n=-\infty}^{\infty} C_n e^{-jn\omega_0 t}$

   b) $\sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$

   c) $\sum_{n=0}^{\infty} C_n e^{-jn\omega_0 t}$

   d) $\sum_{n=0}^{\infty} C_n e^{jn\omega_0 t}$

11. x(t) be the real-valued signal with fundamental period T and Fourier coefficients c_n then Fourier coefficients of $x_e(t)$ are equal to [ ]

   a) $\mathrm{Re}\{c_n\}$

   b) c_n

   c) $\mathrm{Im}\{c_n\}$

   d) $j\mathrm{Im}\{c_n\}$

12. Which of the following signals are periodic? [ ]

   a) f(t)=2cos(10t+1)-sin(4t-1).

   b) f(t)=cos60πt+ sin50πt.

c) f(t)=2u(t)+2Sin2t.

d) f(t)=u(t)-(1/2)

13. For the given periodic function $f(t) = \begin{cases} 2t & \text{for } 0 \le t \le 2 \\ 4 & \text{for } 2 \le t \le 6\ (= T) \end{cases}$ .The coefficient b_n of the continuous Fourier series associated with the given function f (t) can be computed as [ ]

a) − 75.6800

b) − 7.5680

c) − 6.8968

d) − 0.7468

14. For the given continuous time signal $f(t) = \begin{cases} Kt & \text{for } 0 \le t \le T/2 \\ K(T-t) & \text{for } T/2 \le t \le T \end{cases}$ .The coefficient a_n of the continuous Fourier series associated with the given function f (t) can be computed as [ ]

a) KT/2

b) KT

c) 2/KT

e) K/T

15. A periodic signal is defined as $f(t) = \begin{cases} (1-t)(1-t) & \text{for } 0 \le t \le T \\ 0 & \text{otherwise} \end{cases}$ .The coefficient b_n of the continuous Fourier series associated with the given function f (t) can be computed as [ ]

a) 0

b) 1

c) 10

f) 20

16. Which of the following is an “even” function of‘t’? [ ]

a) t^2

b) t^2-4t

c) sin(2t) + 3t

d) t^3 +6

17. A “periodic function” is given by a condition which [ ]

a) Has a period T=2π

b) Satisfies f (t+T) = f(t)

c) Satisfies f (t+T) = −f(t)

d) Has a period T=π

18. The time domain signal for following Fourier series coefficients [ ]

$$c_n = j\delta(n-1) - j\delta(n+1) + \delta(n-3) + \delta(n+3);\ \omega_0 = \pi \text{ is}$$

a) 2cos3πt-sinπt

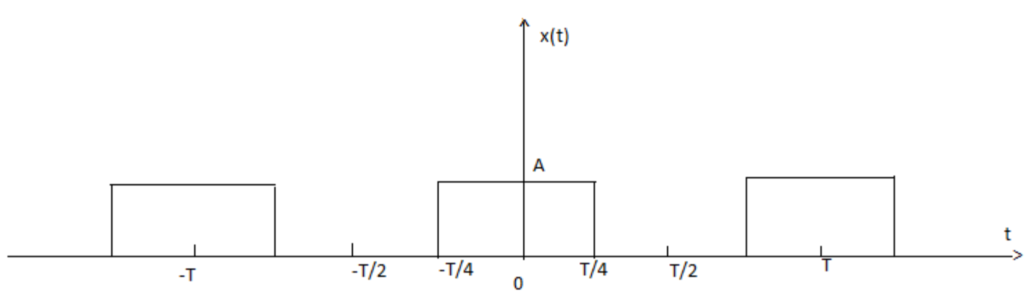
b) 2cos3πt-2sinπt

c) 2cosπt-2sinπt

d) cos3πt-sinπt

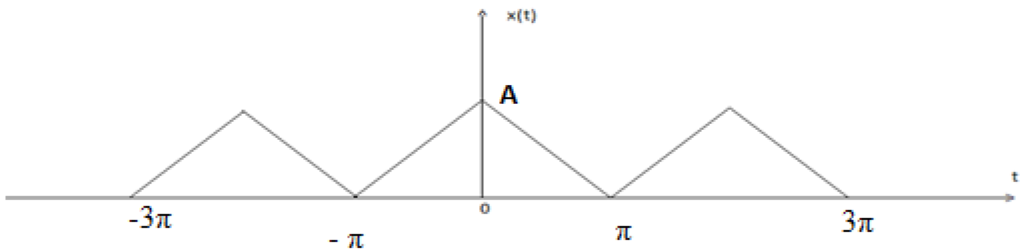
Section-B

1. Obtain the relationship between trigonometric and exponential Fourier series.
2. Explain the concept of complex Fourier spectrum.
3. For the continuous time periodic signal x (t) = 2+cos (2πt/3) +4sin (5πt/3), determine the fundamental frequency ω_0 and the Fourier series coefficients c_n?
4. Obtain the Fourier components of the periodic rectangular wave form given below.

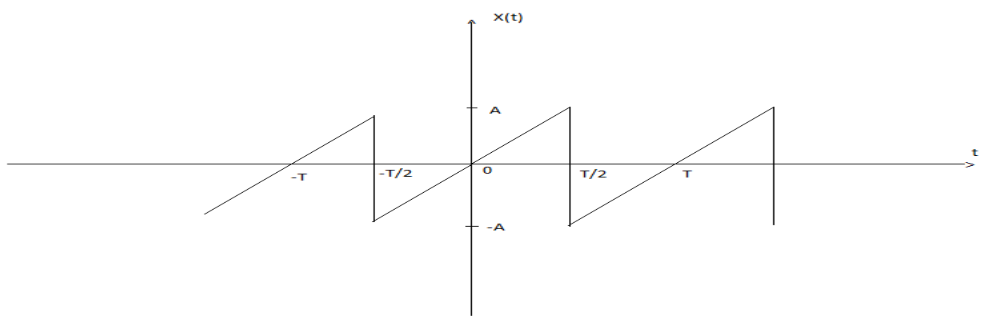


5. Find the Exponential Fourier series and plot frequency spectrum for the full wave rectified sine wave?

6. Find the trigonometric Fourier series for the waveform x(t).



7. Obtain the trigonometric fourier series for the waveform given below.



8. Find the average power of the signal $x(t) = \cos^2(5000\pi t)\sin(20000\pi t)$. If this signal is transmitted through a telephone system which blocks dc and frequencies above 14 kHz, then compute the ratio of received power to transmitted power.

Section-C

1. The Fourier Series of an Odd Periodic function contains only **(GATE-94)**
   a) odd harmonics
   b) even harmonics
   c) cosine terms
   d) sine terms
2. The trigonometric Fourier series for an even function of time does not have **(GATE-96)**
   a) the dc terms
   b) cosine terms
   c) sine terms
   d) odd harmonics terms

3. The trigonometric Fourier series of a Periodic time function can have only

(GATE-98)

a) cosine terms
b) sine terms
c) cosine and sine terms
d) dc and cosine terms

4. The Fourier series of a real periodic function has only **(GATE-09)**

P: cosine terms if it is even

Q: sine terms if it is even

R: cosine terms if it is odd

S: sine terms if it is odd

Which of the above statements are correct?

a) P and S
b) P and R
c) Q and S
d) Q and R

5. The trigonometric Fourier series for an even function does not have the

(GATE-11)

a) the dc terms
b) cosine terms
c) sine terms
d) odd harmonics terms

6. The RMS value of a rectangular wave of period T , having a value of +V for a duration, T_1(<T)and –V for the duration, $T-T_1 = T_2$, equal **(GATE-95)**

a) V
b) $(T_1-T_2)V/T$
c) $V/\sqrt{2}$
d) $T_1{*}V/T_2$

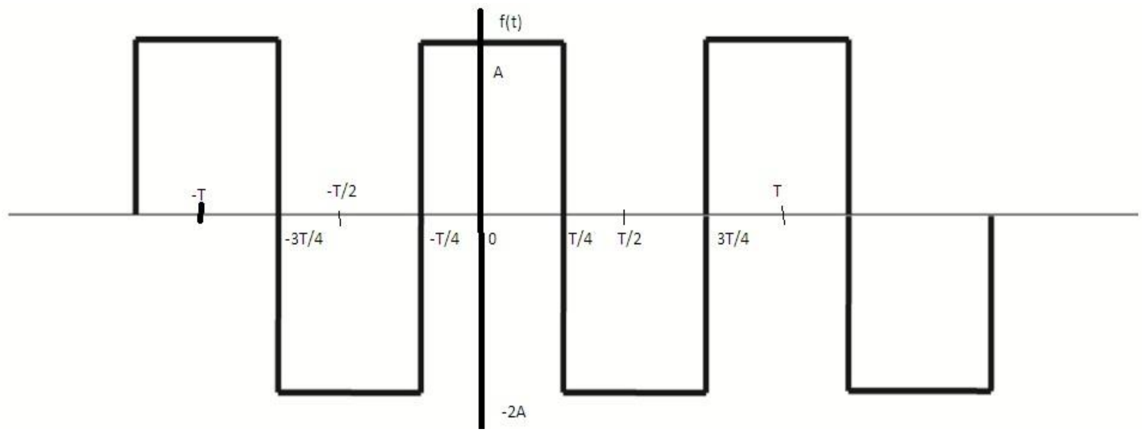
7. A periodic signal x(t) of period T_0 is given by **(GATE-98)**

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T_0/2 \end{cases}$$

The D.C. component of x (t) is

a) T_1/T_0
b) $T_1/(2T_0)$
c) 2T1/T0
d) T_0/T_1

8. Which of the following cannot be the Fourier series expansion of a periodic signal? **(GATE-02)**
   a) x(t)=2 cos t+3 cos 3t
   b) x(t)=2 cos πt+7 cos t
   c) x(t)=cos t+ 0.5
   d) x(t)=2 cos 1.5π t+sin 3.5πt
9. The Fourier series expansion of a real periodic signal with fundamental frequency f_0 is given by, $g_p(t) = \sum_{a=-\infty}^{\infty} cne^{j2\pi f0t}$.It is given that C3 = 3+j5 then, C-3 is **(GATE-03)**
   g) 5+j3
   h) -3-j5
   i) -5+j3
   j) 3-j5
10. Choose the function f(t); -∞<t<+∞, for which a fourier series cannot be defined. **(GATE-05)**
    a) 3 sin(25t)
    b) 4 cos (20t+3)+ 2 sin (10t)
    c) exp(-|t|) sin (25t)
    d) 1
11. A function is given by f(t) = $\sin^2 t$+cos 2t, which of the following is true? **(GATE-09)**
    a) f has frequency components at 0 and 1/2π Hz
    b) f has frequency components at 0 and 1/π Hz
    c) f has frequency components at 1/2π and 1/π Hz
    d) f has frequency components at 1/2π and 1/π Hz
12. The trigonometric Fourier series for the waveform f(t) shown below contains **(GATE-10)**



a) only cosine terms and zero value for the dc components
b) only cosine terms and positive value for the dc components
c) only cosine terms and negative value for the dc components
d) only sine terms and negative value for the dc components

13. A half wave rectifier sinusoidal waveform has a peak voltage of 10V. Its average value and the peak value of the fundamental component are respectively given by: **(GATE-87)**

a) $20/\pi$V, $10/\sqrt{2}$V
b) $10/\pi$V, $10/\sqrt{2}$V
c) $10/\pi$V, 5V
d) $20/\pi$V, 5V

14. Which of the following signals is/are periodic? **(GATE-92)**

e) s(t)=cos2t+cos3t+cos5t.
f) s(t)=exp(j8πt).
g) s(t)=exp(-7t)sin$10\pi t$.
h) s(t)=cos2tcos4t.

15. The Fourier series representation of an impulse train denoted by $S(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_0)$ is given by **(GATE-99)**

a) $\left(\frac{1}{T_0}\right)\sum_{n=-\infty}^{\infty} \exp\left(-j2\pi nt/T_0\right)$
b) $\left(\frac{1}{T_0}\right)\sum_{n=-\infty}^{\infty} \exp\left(-j\pi nt/T_0\right)$
c) $\left(\frac{1}{T_0}\right)\sum_{n=-\infty}^{\infty} \exp\left(j\pi nt/T_0\right)$
d) $\left(\frac{1}{T_0}\right)\sum_{n=-\infty}^{\infty} \exp\left(j2\pi nt/T_0\right)$

UNIT – III

Objective:

- To introduce the concepts of sampling of continuous time signlas.
- To introduce the concept of Fourier transform.

Syllabus: Representation of an arbitrary function over the entire interval: Fourier transform, Existence of Fourier transform, Fourier transform of some useful functions, Fourier transform of periodic function, Properties of Fourier transform, Energy Density spectrum, Parseval's Theorem.
Sampling: Sampling theorem for band limited signals- Explanation, reconstruction of signal from samples, Aliasing, Sampling Techniques-Impulse, Natural and Flat top sampling.

Outcomes:
Students will be able to

- ➢ Perform transformations on signals.
- ➢ Understand the concept of sampling.

Learning Material

- By the use of Fourier Series we can represent any periodic function f(t) over the entire interval as a discrete sum of exponential functions.

Representation of an arbitrary function over the entire interval:

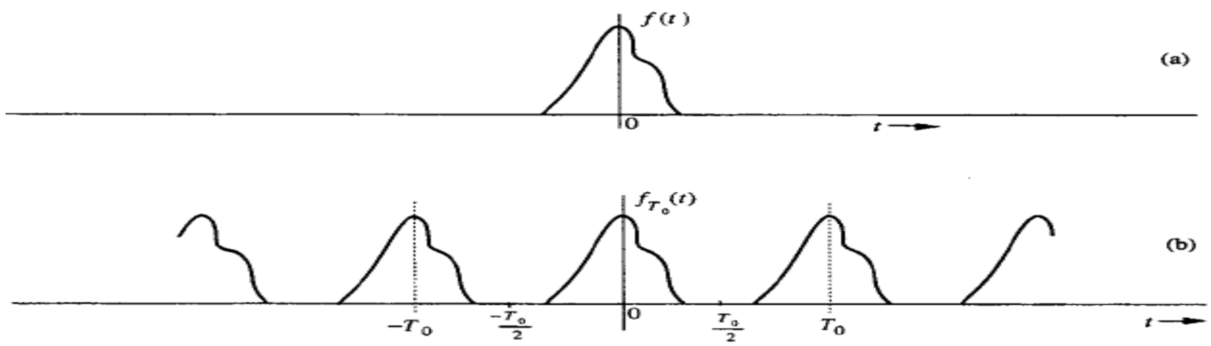


Fig. Construction of a periodic signal by periodic extension of f(t)

- This can be done by constructing a periodic function of period T. So that f(t) represents the first cycle of this periodic waveform.
- If T-> α, then the pulses in periodic function repeat after an infinite interval.

$$lt_{T->\infty} f_T(t) = f(t) \text{------------------------}(1)$$

- The exponential Fourier series of $f_T(t)$ is

$$f_T(t) = \sum_{n=-\infty}^{\infty} F_n e^{jntw_0}$$

Where $$w_0 = \frac{2\pi}{T}$$

$$F_n = \frac{1}{T}\int_{\frac{-T}{2}}^{\frac{T}{2}} f_T(t) e^{jntw_0} dt \text{------------------------}(2)$$

- Let us consider $nw_0 = w_n$ then F_n is the function of w_n, and we shall denote F_n by $F_n(w_n)$

  From Equation (2)

$$TF_n = \int_{\frac{-T}{2}}^{\frac{T}{2}} f_T(t)e^{jtw_n}\,dt$$

Let $TF_n(w_n) = F(w_n)$ --(3)

Then $f_T(t) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} F(w_n)w_0 e^{jtw_n}$ --(4)

- From the above equation $f_T(t)$ is expressed as a sum of exponential signals of frequencies $w_1, w_2, \ldots\ldots\ldots\ldots w_n$. Let us represent the above equation (4) in graphical representation

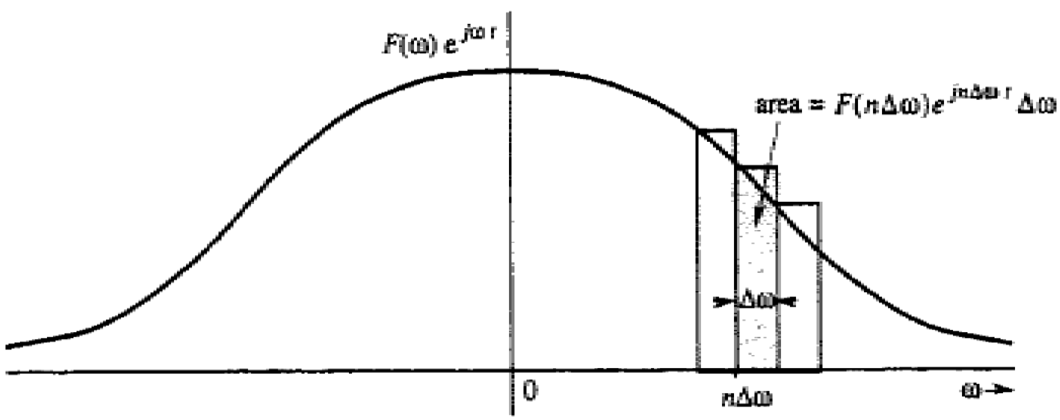


Fig. The Fourier series become Fourier integral as limit $T -> \infty$

- The discrete sum in equation (4) becomes the integral or the area under this curve. The curve now is continuous function of w and is given by $F(w)e^{jwt}$.

- Also $T -> \infty$**,** the function $f_T(t)$ -> f(t)

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(w)e^{jwt}dw \text{-----------------------------------(5)}$$

Where $F(w) = \int_{-\infty}^{\infty} f(t)e^{-jwt}dt$ --------------------------------------(6)

<u>Existence of the Fourier Transform:</u>

The conditions for a function to have Fourier transform, called dirichilet's conditions

- f(t) is absolutely integrable over the interval $-\infty$ to ∞ that is $\int_{-\infty}^{\infty} f(t)dt < \infty$
- f(t) has a finite number of discontinuities in every finite interval.
- f(t) has a finite number of maxima and minima every finite interval.

Fourier transform of some useful functions:

	$f(t)$	$F(\omega)$	
1	$e^{-at}u(t)$	$\dfrac{1}{a+j\omega}$	$a>0$
2	$e^{at}u(-t)$	$\dfrac{1}{a-j\omega}$	$a>0$
3	$e^{-a\|t\|}$	$\dfrac{2a}{a^2+\omega^2}$	$a>0$
4	$te^{-at}u(t)$	$\dfrac{1}{(a+j\omega)^2}$	$a>0$
5	$t^n e^{-at}u(t)$	$\dfrac{n!}{(a+j\omega)^{n+1}}$	$a>0$
6	$\delta(t)$	1	
7	1	$2\pi\delta(\omega)$	
8	$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$	
9	$\cos\omega_0 t$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$	
10	$\sin\omega_0 t$	$j\pi[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)]$	
11	$u(t)$	$\pi\delta(\omega)+\frac{1}{j\omega}$	
12	$\operatorname{sgn} t$	$\frac{2}{j\omega}$	
13	$\cos\omega_0 t\, u(t)$	$\frac{\pi}{2}[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]+\frac{j\omega}{\omega_0^2-\omega^2}$	
14	$\sin\omega_0 t\, u(t)$	$\frac{\pi}{2j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]+\frac{\omega_0}{\omega_0^2-\omega^2}$	
15	$e^{-at}\sin\omega_0 t\, u(t)$	$\frac{\omega_0}{(a+j\omega)^2+\omega_0^2}$	$a>0$
16	$e^{-at}\cos\omega_0 t\, u(t)$	$\frac{a+j\omega}{(a+j\omega)^2+\omega_0^2}$	$a>0$
17	$\operatorname{rect}\left(\frac{t}{\tau}\right)$	$\tau\operatorname{sinc}\left(\frac{\omega\tau}{2}\right)$	
18	$\frac{W}{\pi}\operatorname{sinc}(Wt)$	$\operatorname{rect}\left(\frac{\omega}{2W}\right)$	
19	$\Delta\left(\frac{t}{\tau}\right)$	$\frac{\tau}{2}\operatorname{sinc}^2\left(\frac{\omega\tau}{4}\right)$	
20	$\frac{W}{2\pi}\operatorname{sinc}^2\left(\frac{Wt}{2}\right)$	$\Delta\left(\frac{\omega}{2W}\right)$	
21	$\displaystyle\sum_{n=-\infty}^{\infty}\delta(t-nT)$	$\omega_0\displaystyle\sum_{n=-\infty}^{\infty}\delta(\omega-n\omega_0)$	$\omega_0=\frac{2\pi}{T}$
22	$e^{-t^2/2\sigma^2}$	$\sigma\sqrt{2\pi}e^{-\sigma^2\omega^2/2}$	

Properties of continuous time Fourier transform:

1. Linearity property: If

$$f_1(t) \Leftrightarrow F_1(w) \qquad \text{and} \qquad f_2(t) \Leftrightarrow F_2(w) \text{ then}$$

$$af_1(t) + bf_2(t) <=> aF_1(w) + bF_2(w)$$

2. Time shifting property: If

$$f(t) \Leftrightarrow F(w) \text{ then}$$

$$f(t - to) \Leftrightarrow F(w) \quad e^{-i\omega to}$$

3. Frequency shifting property: If

$$f(t) \Leftrightarrow F(w) \quad \text{then}$$

$$f(t)e^{-i\omega ot} \Leftrightarrow F(w - w_o)$$

4. Time Reversal property: If

$$f(t) \Leftrightarrow F(w) \quad \text{then}$$

$$f(-t) \Leftrightarrow F(-w)$$

5. Time Scaling property: If

$$f(t) \Leftrightarrow F(w) \quad \text{then}$$

$$f(at) \Leftrightarrow \frac{1}{|a|} F\left(\frac{w}{a}\right)$$

6. Time Differentiation Property: If

$$f(t) \Leftrightarrow F(w) \quad \text{then}$$

$$\frac{d}{dt} f(t) \Leftrightarrow jwF(w)$$

7. Time Integration Property: If

$$f(t) \Leftrightarrow F(w) \quad \text{then}$$

$$\int_{-\infty}^{t} f(\tau)d\tau \Leftrightarrow \frac{F(w)}{jW} + \pi F(0)\delta(w)$$

8. Time Convolution Property: If

$$f_1(t) \Leftrightarrow F_1(w) \text{ and } f_2(t) \Leftrightarrow F_2(w) \text{ then}$$

$$f_1(t) * f_2(t) \Leftrightarrow F_1(w)F_2(w)$$

9. Frequency Convolution Property: If

$$f_1(t) \Leftrightarrow F_1(w) \text{ and } f_2(t) \Leftrightarrow F_2(w) \text{ then}$$

$$f_1(t)f_2(t) \Leftrightarrow \frac{1}{2\pi} F_1(w) * F_2(w)$$

10. Duality Property: If

$$f(t) \Leftrightarrow F(w) \quad \text{then}$$

$$F(t) \Leftrightarrow 2\pi f(-w)$$

11. Modulation Property: If

$$f(t) \Leftrightarrow F(w) \quad \text{then}$$

$$f(t) cos w_o t \Leftrightarrow \frac{1}{2}[F(w - w_0) + F(w + w_o)$$

12. Parseval's Relation: If

$$f_1(t) \Leftrightarrow F_1(w) \text{ and } f_2(t) \Leftrightarrow F_2(w) \text{ then}$$

$$\int_{-\infty}^{\infty} f_1(t) f_2^*(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} F_1(w) F_2^*(w) dw$$

Fourier Transform of a periodic function:

Let f(t) is a periodic function of period T. The Fourier transform of periodic function is

$$\Im[f(t)] = \sum_{-\infty}^{\infty} F_n \Im[e^{jntw_t}]$$

$$F(w) = \sum_{n=-\infty}^{\infty} F_n 2\pi \delta(w - nw_0)$$

Sampling Theorem:

Statement: A band limited signal f(t) with F(w)=0 for $|w| \geq w_m$ can be represented into and uniquely determined from its samples f(nT) if the sampling frequency $f_s \geq 2 f_m$, where f_m is the highest frequency component present in it. That is for signal recovery, the sampling frequency must be at least twice the highest frequency present in the signal.

- Let f(t) is a continuous time band limited signal to be sampled which has no spectral components above f_m.

- Let $\delta_T(t)$ is an impulse train which samples at a rate of fs Hz

- Then the sampled signal

$$f(t)\delta_T(t) = \sum_n f(nT)\delta(t - nT)$$

- The Fourier transform of sampled signal is

$$\bar{F}(w) = \frac{1}{T}\sum_{n=-\infty}^{\infty} F(w - nwo)$$

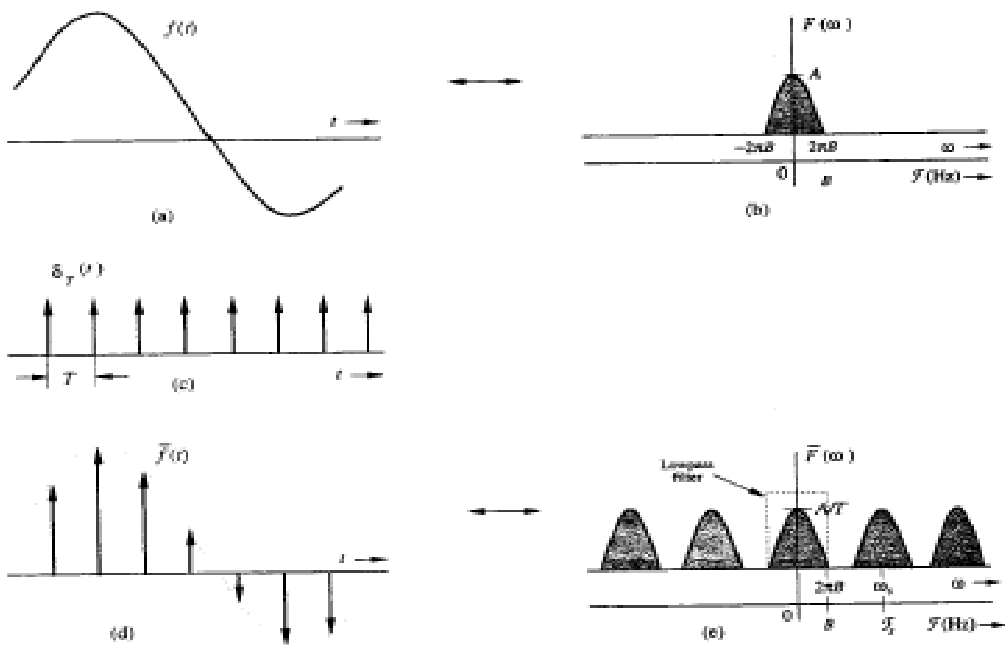


Fig. sampled signal and its Fourier spectrum.

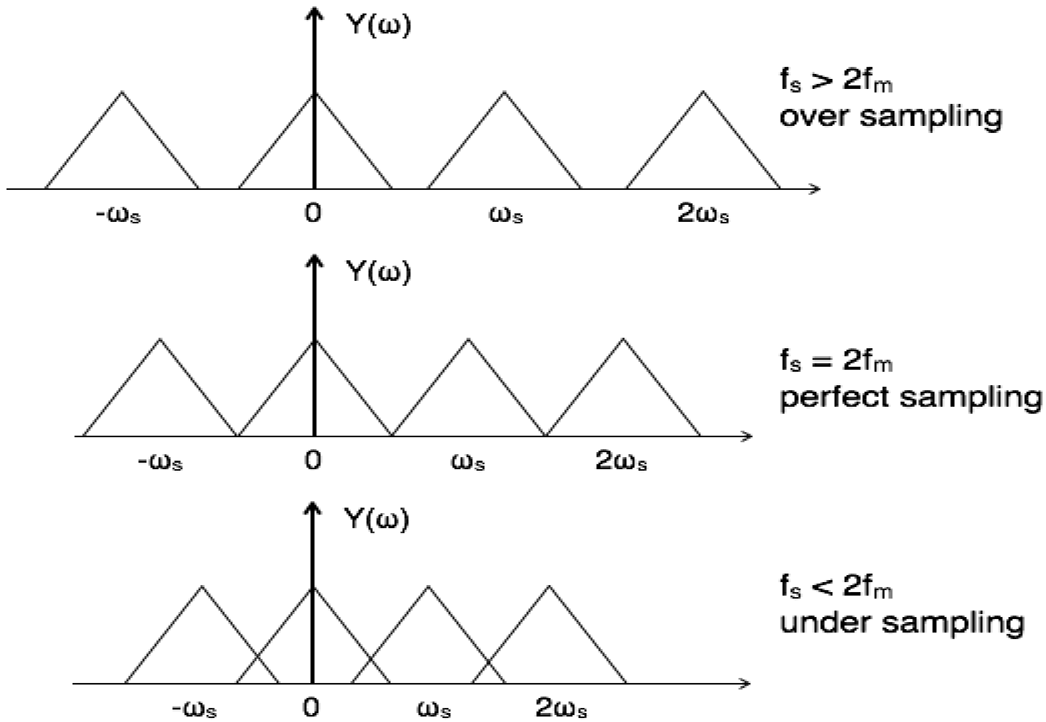


Fig. Effect of under sampling and over sampling.

- Nyquist rate of sampling is the theoretical minimum sampling rate at which a signal can be sampled and still be reconstructed from its samples without any distortion.
- A signal sampled at greater than Nyquist rate is said to be over sampled and signal sampled is less than its Nyquist rate is said to be under sampled.

<u>Reconstruction of signal from it's samples:</u>

- The original function can be recovered by passing the sampled function through a low pass filter with a cutoff frequency f_m
- This can be obtained by multiplying the sampled signal by a gate function

$$h(t) = rect\left(\frac{t}{T}\right) = rect(2Bt)$$

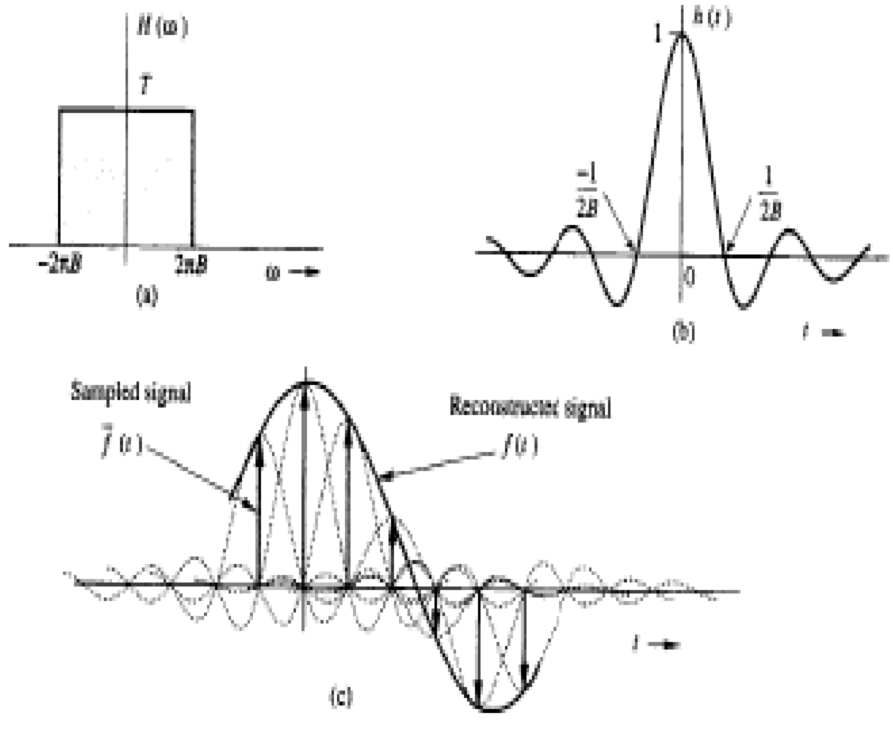
and

$$H(w) = Tsinc\left(\frac{wT}{2}\right) = \frac{1}{2B}sinc(\frac{w}{4B})$$

The resultant output of the filter is

$$f(t) = \sum_k f(kT)h(t-kT)$$

$$= \sum_k f(kT)sinc(2\pi Bt - k\pi)$$

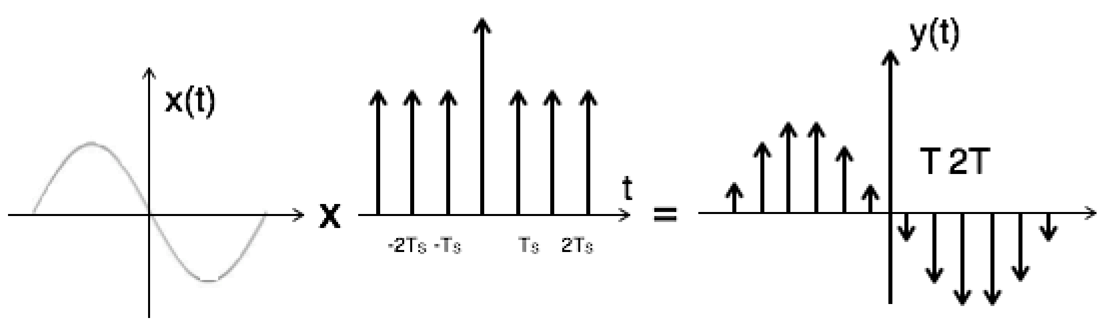


Sampling techniques:

There are three types of sampling techniques

Impulse sampling:

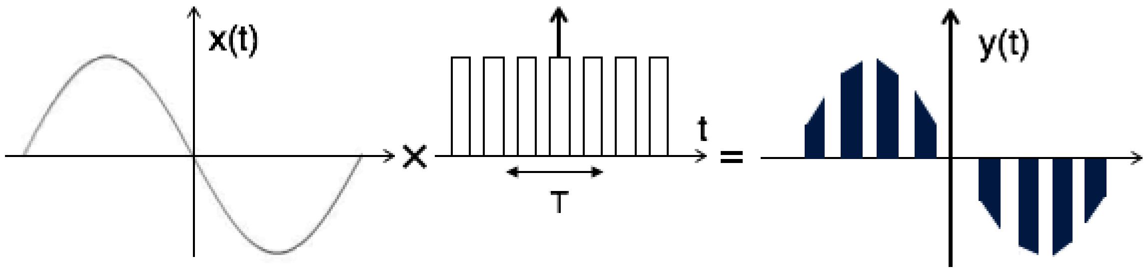
- Impulse sampling can be performed by multiplying input signal x(t) with impulse train $\sum_{n=-\infty}^{\infty}\delta(t-nT)$ of period 'T'. Here, the amplitude of impulse changes with respect to amplitude of input signal x(t). The output of sampler is given by



$$y(t)= x(t) \times \sum_{n=-\infty}^{\infty}\delta(t-nT)$$

Natural Sampling:

- Natural sampling is similar to impulse sampling, except the impulse train is replaced by pulse train of period T. i.e. you multiply input signal x(t) to pulse train $\sum_{n=-\infty}^{\infty} p(t-nT)$ as shown below



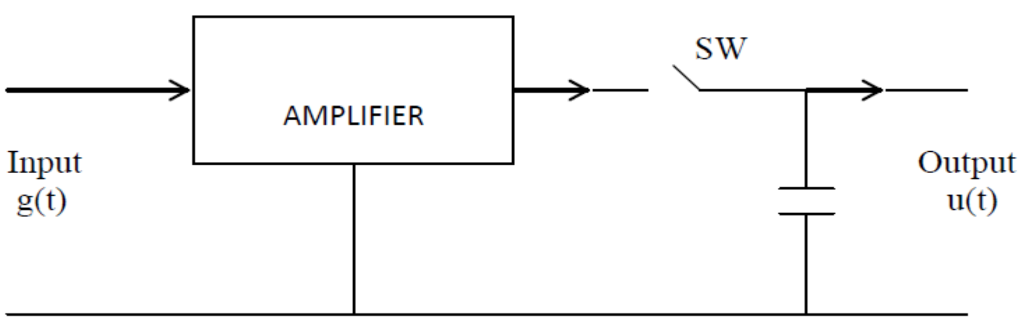
$$y(t) = x(t) \times \sum_{n=-\infty}^{\infty} p(t-nT)$$

Flat top sampling:

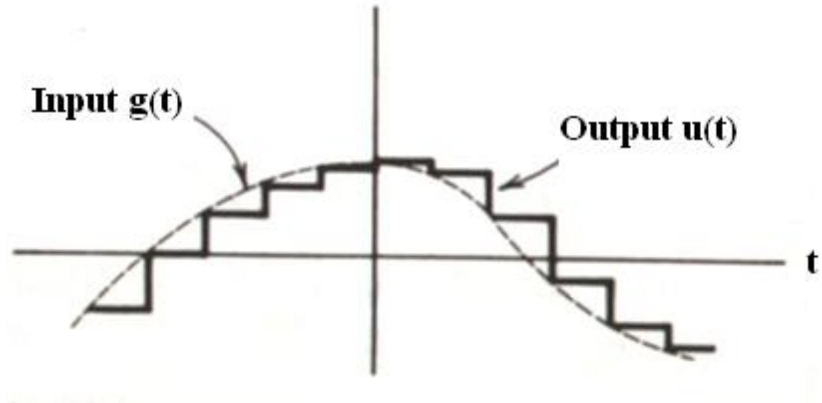
- The Sample-and-Hold circuit consists of an amplifier of unity gain and low output impedance, a switch and a capacitor; it is assumed that the load impedance is large.

- The switch is timed to close only for the small duration of each sampling pulse, during which time the capacitor charges up to a voltage level equal to that of the input sample.

- When the switch is open , the capacitor retains the voltage level until the next closure of the switch. Thus the sample-and-hold circuit produces an output waveform that represents a staircase interpolation of the original analog signal.



a) Sample and Hold Circuit



b) Sampled signal

UNIT-IV

LTI Systems: Signal Transmission through LTI Systems

Learning Objectives:

- To know the concept of LTI system
- To understand the filter characteristics of LTI system
- To understand the concept of Causality and Paley-Wiener Criterion for Physical Realization

Learning outcomes:

After the completion of this chapter, student will be able to

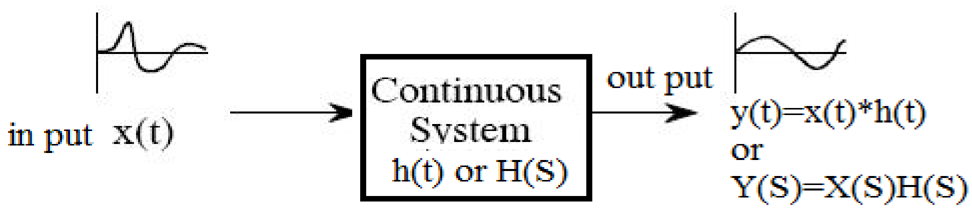
- Define and classify systems
- Find the transfer function of a system
- Define the cutoff frequency of filter
- Design a physically realizable system

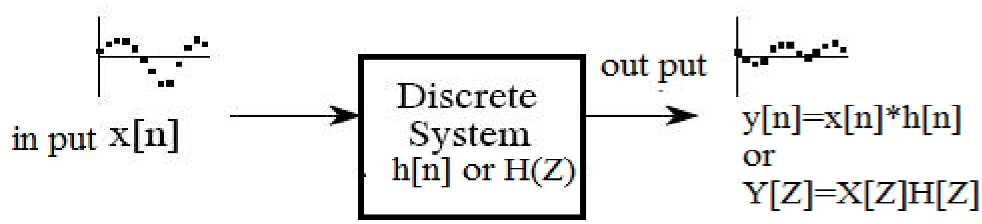
Syllabus: LTI Systems

Properties of systems, Linear Time Invariant (LTI) system, Response of LTI system-Convolution Integral, Graphical interpretation; Properties of LTI system, Transfer function and Frequency Response of LTI system.

Signal Transmission through LTI Systems: Filter characteristics of LTI Systems, distortion less transmission through LTI system, Signal bandwidth, System bandwidth, Ideal LP, HPF and BPF characteristics, Causality and Physical reliability- Paley-Wiener Criterion, Relationship between bandwidth and rise-time.

System: A system may be define as "A set of components that are connected together to perform the particular task".





Systems are classified into the following categories:

- Liner and Non-liner Systems
- Time Variant and Time Invariant Systems
- Liner Time variant and Liner Time invariant systems
- Static and Dynamic Systems
- Causal and Non-causal Systems
- Invertible and Non-Invertible Systems

- Stable and Unstable Systems

Liner and Non-liner Systems:

- ✓ *A system is said to be linear when it satisfies superposition and homogenate principles.*

Consider two systems with inputs as $x_1(t)$, $x_2(t)$, and outputs as $y_1(t)$, $y_2(t)$ respectively. Then, according to the superposition and homogenate principles,

$$T[a_1 x_1(t) + a_2 x_2(t)] = a_1 T[x_1(t)] + a_2 T[x_2(t)]$$
$$T[a_1 x_1(t) + a_2 x_2(t)] = a_1 y_1(t) + a_2 y_2(t)$$

From the above expression, is clear that response of overall system is equal to response of individual system.

Example:

$Y(t) = x^2(t)$

When input is $x_1(t)$

Then output is $y_1(t) = T[x_1(t)] = x_1^2(t)$

When input is $x_2(t)$

Then output is $y_2(t) = T[x_2(t)] = x_2^2(t)$

When input is $a_1 x_1(t) + a_2 x_2(t)$

The output is $T[a_1 x_1(t) + a_2 x_2(t)] = [a_1 x_1(t) + a_2 x_2(t)]^2$

Which is not equal to $a_1 y_1(t) + a_2 y_2(t)$. Hence the system is said to be non linear.

Time Variant and Time Invariant Systems

- ✓ *A system is said to be time variant if its input and output characteristics vary with time. Otherwise, the system is considered as time invariant.*

The condition for time invariant system is:

$$y(n, t) = y(n-t)$$

The condition for time variant system is:

$$y(n, t) \neq y(n-t)$$

Where $y(n, t) = T[x(n-t)]$ = input change ; $y(n-t)$ = output change

Example:

$$y(n) = x(-n)$$

then $y(n, t) = T[x(n-t)] = x(-n-t)$

and $y(n-t) = x(-(n-t)) = x(-n + t)$

$\therefore$ $y(n, t) \neq y(n-t)$. Hence, the system is time variant.

Liner Time variant (LTV) and Liner Time Invariant (LTI) Systems

- ✓ *If a system is both liner and time variant, then it is called liner time variant (LTV) system.*
- ✓ *If a system is both liner and time Invariant then that system is called liner time invariant (LTI) system.*

Static and Dynamic Systems

- ✓ *Static system is memory-less whereas dynamic system is a memory system.*

Example : $y(t) = 2x(t)$

For present value t=0, the system output is y(0) = 2x(0). Here, the output is only dependent upon present input. Hence the system is memory less or static.

Causal and Non-Causal Systems

- ✓ *A system is said to be causal if its output depends upon present and past inputs, and does not depend upon future input.*
- ✓ *For non causal system, the output depends upon future inputs also.*

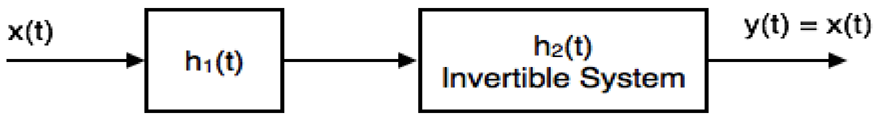
Example 1: y(n) = 2 x(t) + 3 x(t-3)

For present value t=1, the system output is y(1) = 2x(1) + 3x(-2).

Here, the system output only depends upon present and past inputs. Hence, the system is causal.

Invertible and Non-Invertible systems

- ✓ *A system is said to invertible if the input of the system appears at the output.*



$$Y(S) = X(S)\, H_1(S)\, H_2(S)$$

$$= X(S)\, H_1(S) \cdot 1/(H_1(S)) \qquad \text{since } H_2(S) = 1/(\,H_1(S)\,)$$

$$\therefore Y(S) = X(S) \quad \text{or} \quad y(t) = x(t)$$

Hence, the system is invertible.

If $y(t) \neq x(t)$, then the system is said to be non-invertible.

Stable and Unstable Systems

- ✓ *The system is said to be stable only when the output is bounded for bounded input. For a bounded input, if the output is unbounded in the system then it is said to be unstable.*

Note: For a bounded signal, amplitude is finite.

Example 1:

$y(t) = x^2(t)$

Let the input is u(t) (unit step bounded input)

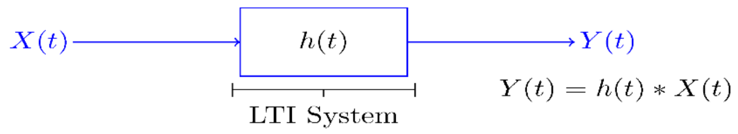
then the output

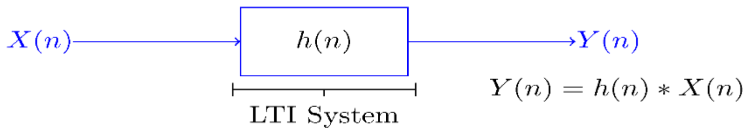
$y(t) = u^2(t) = u(t)$ = bounded output.

Hence, the system is stable.

Liner Time Invariant (LTI) Systems

If a system is both liner and time Invariant then that system is called liner time invariant (LTI) system





Response of Basic LTI System

For linear time-invariant (LTI) systems the convolution integral can be used to obtain the output (response)from the input and the system impulse response

Convolution Integral

$$\boxed{y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t)*h(t)}$$

h(t) is called impulse response of the system

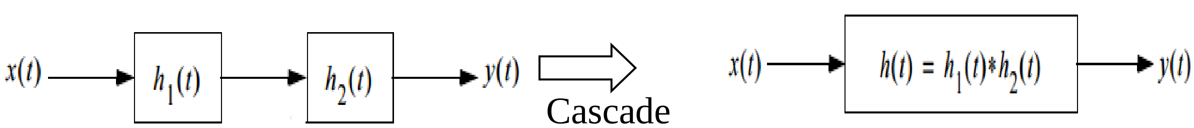
Impulse response: it is obtained at the output of a system by applying impulse signal (i.e. δ (t))
When input x(t) = δ (t) then y(t) = h(t).

Properties of LTI Systems :

- ✓ LTI system satisfies homogeneity and liner properties.

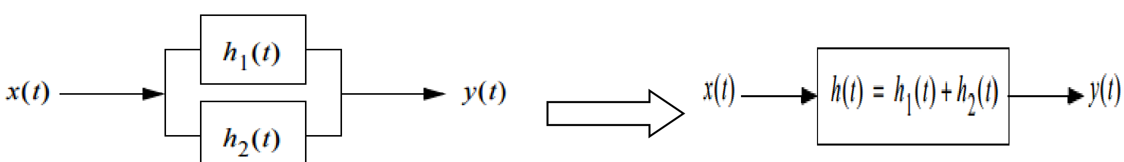
Cascade and Parallel Connections:

For a cascade of two LTI systems having impulse responses h1(t) and h2(t) respectively, the impulse response of the cascade is the convolution of the impulse responses



$$\boxed{h_{\text{cascade}}(t) = h_1(t)*h_2(t)}$$

For two systems connected in parallel, the impulse response is the sum of the impulse responses

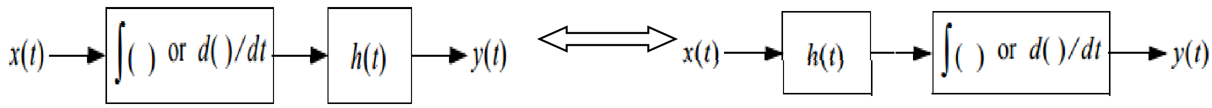


Parallel

$$\boxed{h_{\text{parallel}}(t) = h_1(t) + h_2(t)}$$

Differentiation and Integration of Convolution:

Performing differentiation or integration before a signal enters and LTI system, gives the same result as performing the differentiation or integration after the signal passes through the system



<u>The Frequency Response Function for LTI Systems</u> :

The output of an LTI system can be given in terms of the convolution integral

$$y(t) = h(t)*x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

Where we recall that is the (unit) impulse response of a system
We choose to start the analysis with a single complex sinusoid

$$x(t) = Ae^{j\phi}e^{j\omega t}, -\infty < \omega < \infty$$

the output is

$$\begin{aligned} y(t) &= h(t)*(Ae^{j\phi}e^{j\omega t}) \\ &= \int_{-\infty}^{\infty} h(\tau)Ae^{j\phi}e^{j\omega(t-\tau)}d\tau \\ &= \left(\int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau\right)Ae^{j\phi}e^{j\omega t} \\ &= H(j\omega)Ae^{j\phi}e^{j\omega t} \end{aligned}$$

We have thus defined the frequency response of an LTI system as

$$\boxed{H(j\omega) = \int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau}$$

<u>Transfer function of LTI system:</u>

It may be defined as the ratio Fourier or Laplace transform of output to Fourier or Laplace transform of input function

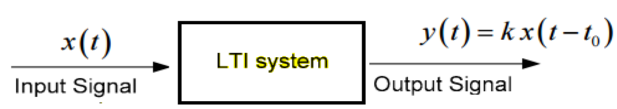
$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)}$$

<u>Distortion less transmission:</u>

The output signal of system is an exact replica of the input signal except for two minor modifications.

- A possible scaling of amplitude
- A constant time delay

A signal x(t) is transmitted through a system without distortion if the output signal is defined by y(t)



The constant 'k' represents a change in amplitude and constant 't_0' indicate a delay in transmission

Using time shifting property

$$Y(j\omega) = k\ X(j\omega)\ e^{-j\omega t_0}$$

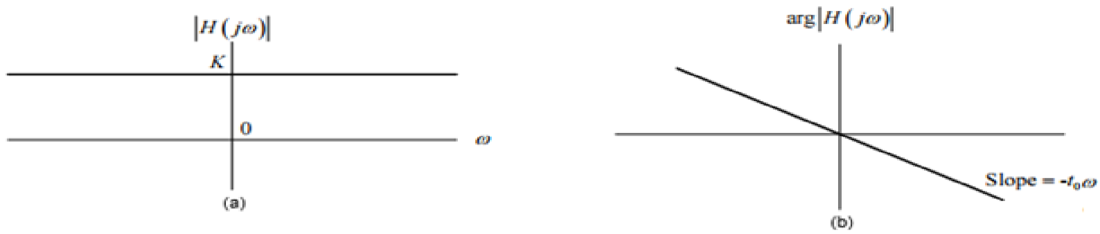
The frequency response of and the impulse response of distortion less system is

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = k\ e^{-j\omega t_0}$$

Hence, Conditions for distortion less transmission of CT LTI system with transfer function H(jω) is, its magnitude response and phase shift for all frequencies are

$$|H(j\omega)| = k \quad \& \quad \angle\ H(j\omega) = -\ \omega\ t_0$$

Frequency response for distortion less transmission through a linear time-invariant system



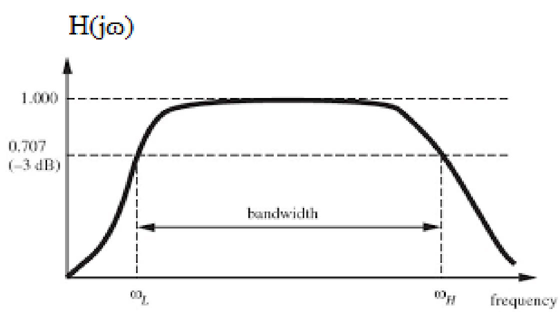
Magnitude response   Phase response

Signal Band width :

An energy signal is having its spectral components from -∞ to +∞. A band of frequencies that contain most of the signal energy is known as the signal Band width. It is selected such that it contains 95% of total energy.

System Bandwidth:

A system is specified by magnitude of transfer function (i.e | H(ω) | or |H(S)|) . System bandwidth may be defined as the range of frequencies over which the magnitude of | H(jω) | remains 0.707 times of its value at mid band



Filter characteristics of LTI system

For LTI system the output obtained by

y(t)= x(t)*h(t) or Y(S)=X(S)H(S) or Y(ω)=X(ω) H(ω)

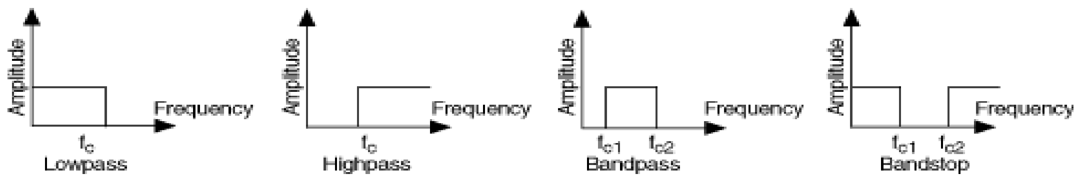
- The transfer function H(ω) will modifies the spectral components of input
- So the system act as a filter for various frequency components

Characteristics of an Ideal Filter:

Ideal filters allow a specified frequency range of interest to pass through while attenuating a specified unwanted frequency range. The following filter classifications are based on the frequency range a filter passes or blocks:

- Low pass filters pass low frequencies and attenuate high frequencies.
- High pass filters pass high frequencies and attenuate low frequencies.
- Band pass filters pass a certain band of frequencies.
- Band stop filters attenuate a certain band of frequencies.

The following figure shows the ideal frequency response of each of the preceding filter types.



In the previous figure, the filters exhibit the following behavior:

- The low pass filter passes all frequencies below f_c.
- The high pass filter passes all frequencies above f_c.
- The band pass filter passes all frequencies between f_{c1} and f_{c2}.
- The band stop filter attenuates all frequencies between f_{c1} and f_{c2}.

The frequency points f_c , f_{c1}, and f_{c2} specify the cut-off frequencies for the different filters.

Ideal LPF: The transfer function of an ideal LPF is given by

$$\begin{aligned} |H(\omega)| &= 1 \quad \text{for } |\omega| < \omega_c \\ &= 0 \quad \text{for } |\omega| > \omega_c \end{aligned}$$

Ideal HPF: The transfer function of an ideal LPF is given by

$$\begin{aligned} |H(\omega)| &= 0 \quad \text{for } |\omega| < \omega_c \\ &= 1 \quad \text{for } |\omega| > \omega_c \end{aligned}$$

Ideal BPF: The transfer function of an ideal LPF is given by

$$\begin{aligned} |H(\omega)| &= 1 \quad \text{for } |\omega_1| < \omega < |\omega_2| \\ &= 0 \quad \text{else} \end{aligned}$$

Ideal BSF: The transfer function of an ideal LPF is given by

$$\begin{aligned} |H(\omega)| &= 0 \quad \text{for } |\omega_1| < \omega < |\omega_2| \\ &= 1 \quad \text{else} \end{aligned}$$

Causality and Paley-Wiener Criterion for Physical Realization:

An LTI system said to be causal, if it has zero impulse response for 't' less than zero i.e.

$$h(t) = 0 \qquad \text{for } t < 0$$

Such systems are physically realizable. This is time domain criterion. In frequency domain, the necessary and sufficient condition for a magnitude function H(ω) to be physically realizable is

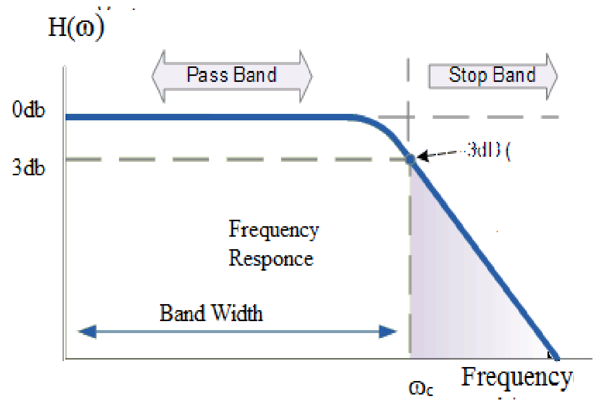
$$\int_{-\infty}^{+\infty} \frac{ln|H(\omega)|}{1 + x^2}$$

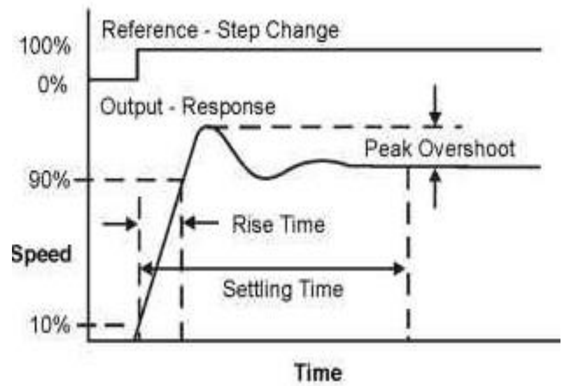
The conclusion are:

1. Ideal filters are not physically realizable.
2. Realizable function magnitude characteristics cannot have total attenuation

Relationship between Bandwidth and Rise Time:

The Rise time (t r is defined as the time required to for response to reach from 0% to 100 % of its final value). The band width of system is defined it is the range of frequencies where the band of frequencies are have its gain (transfer function H(ω)) 0.707 of its maximum value) i.e. the 3dB points deference





Bandwidth is related with Raise time by

$$\boxed{t_r = \frac{\Pi}{BW}}$$

Unit V

CORRELATION OF CONTINUOUS TIME SIGNALS

Objectives:

To familiarize the concepts of various types of correlation, properties and convolution.

Syllabus:

Cross correlation and auto correlation of continuous time signals, relation between convolution and correlation, properties of cross correlation and autocorrelation, power density spectrum, relation between auto correlation function and energy/power spectral density function.

Outcomes:

Students will be able to

- Classify various types of correlations.
- Identify power-energy density spectrum.
- Determine cross correlation and auto correlation.
- Distinguish between correlation and convolution.

Concept of Convolution:

- Convolution is a mathematical operation which can be performing on two signals $x_1(t)$ and $x_2(t)$ to produce a third signal which is typically viewed as the modified version of one of the original signals.
- A convolution is an integral that express the overlap of one signal $x_2(t)$ as it is shifted over another signal $x_1(t)$.

- Convolution of two signals $x_1(t)$ and $x_2(t)$ over a finite range *[0 → t]* can be defined as

$$[x_1 * x_2](t) = \int_{-\infty}^{\infty} x_1(T)\, x_2(t-T)dT$$

Here the symbol $[x_1 * x_2](t)$denotes the convolution of $x_1(t)$ and$x_2(t)$. Convolution is

More often taken over an infinite range like,

$$[x_1 * x_2](t) = y(t) = \int_{-\infty}^{\infty} x_1(T)\, x_2(t-T)dT$$

- The convolution of two discrete time signals $x_1(n)$ and $x_2(n)$ over an infinite range can be defined as

$$[x_1 * x_2](n) = y(n) = \sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k]$$

Convolution properties:

- There are some important properties of convolution that perform on continuous time signal which we have listed below. The commutative, associative, distributive properties are given below.
- Commutative: $x_1(t) * x_2(t) = x_2(t) * x_1(t)$
- Associative: $[x_1(t) * x_2(t)] * x_3(t) = x_1(t) * [x_2(t) * x_3(t)]$
- Distributive: $x_1(t) * [x_2(t) + x_3(t)] = [x_1(t) * x_2(t) + x_1(t) * x_3(t)]$

EDS (Energy Density Spectrum)

- **Spectral density**: It is the distribution of power or energy of a signal per unit band width as a function of frequency.

- **Energy density**: signals with finite energy 0< $E < \infty$ *and* $P = 0$ *called energy signal*
- **power density:** signals with finite average power 0< $P < \infty$ *and* $E = \infty$ *called power or periodic signal*
- **Normalized energy**: Energy dissipated by a voltage signal applied across 1 ohm resistor

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

- **Parseval's theorem for energy signals:** Defines the energy of a signal in terms of FT

$$E = \int_{-\infty}^{\infty} |x(f)|^2 \, df$$

- **Energy density Spectrum:** It is the distribution of energy of a signal in the frequency domain called ESD. $\Psi(f) = |x(f)|^2$

PDS (Power Density Spectrum)

- **Power signal:** These are the signals with infinite energy, but with finite average power.
- **Average power:** power dissipated by a voltage signal applied across 1 ohm resistor. Mathematically $P = \lim_{t\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2 \, dt$
- **Parseval's theorem:** Defines the power of a signal in terms of the harmonic components present in the signal. Mathematically $P=\sum_{-\infty}^{\infty} |F_n|^2$ We have the exponential Fourier series

$$X(t) = \sum_{-\infty}^{\infty} F_{ne^{-j\omega 0nt}}$$

$$P = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2 \, dt = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \sum_{-\infty}^{\infty} F_n e^{-j\omega 0nt} x^*(t) dt$$

$$= \frac{1}{T}\sum_{-\infty}^{\infty} F_n \int_{-\frac{T}{2}}^{\frac{T}{2}} x^*(t)\, e^{-j\omega 0nt} dt = \sum_{-\infty}^{\infty} F_n \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x^*(t)\, e^{-j\omega 0nt} dt$$

$$= \sum_{-\infty}^{\infty} |F_n \, F_{n|^*} = P = \sum_{-\infty}^{\infty} |F_n|^2$$

Called parseval's power theorem.

- **Power Density Spectrum:** The distribution of average power of the signal in the frequency domain is called power spectral density (PSD)

Cross-correlation and autocorrelation of continuous time signals:

Correlation:

- Correlation is a measure of similarity between two signals. The general formula for correlation is

$$\int_{-\infty}^{\infty} x_1(t)x_2(t-\tau)dt = \int_{\infty}^{-\infty} x_1(t)x_2(t-\tau)dt$$

- There are two types of correlation:
  - Auto correlation
  - Cross correlation

Auto Correlation Function:

- It is defined as correlation of a signal with itself. Autocorrelation function is a measure of similarity, regularity, match or coherence between a signal & its time delayed version. It is represented with $R_{xx}(\tau)$.

Consider a signals x(t). The autocorrelation function of x (t) with its time delayed version is given by

$$R_{xx}(\tau) \text{ or } R(\tau) = \lim_{t\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)x^*(t-\tau)\, dt$$

This is defined separately for energy and power Signals.

- Autocorrelation foe energy signals

  Consider energy signal x(t) then the autocorrelation ot this signal is obtained by integrating the product of the x(t) and delayed version of its complex conjugate

$$R_{xx}(\tau) \text{ or } R(\tau) = \int_{-\infty}^{\infty} x(t)x^*(t-\tau)\, dt$$

τ is called the time delay parameter

If signals x(t) is shifted by the same period,

$R_{xy}(\tau)$ or $R(\tau)= \int_{-\infty}^{\infty} x(t+\tau)x^*(t)\,dt$

Properties of Autocorrelation Function of Energy Signal:

- Autocorrelation exhibits conjugate symmetry i.e. $R(\tau) = R^*(-\tau)$

  Proof: Autocorrelation function of energy signal is given by

  taking complex conjugate

  $R^*(\tau)= \int_{-\infty}^{\infty} x(t-\tau)x^*(t)\,dt$ $R_{xy}(\tau)$ or $R(\tau)= \int_{-\infty}^{\infty} x(t)x^*(t-\tau)\,dt$

  $R^*(-\tau)= \int_{-\infty}^{\infty} x(t+\tau)x^*(t)\,dt = R(\tau)$

  $R(\tau) = R^*(-\tau)$

- The value autocorrelation function of energy signal is given at origin i.e. at $\tau=0$ is equal to total energy of that signal, which is given as:

$$R(0) = E = \int_{-\infty}^{\infty} |x(t)|^2\,dt$$

  Proof: $R_{xx}(\tau)$ or $R(\tau)= \int_{-\infty}^{\infty} x(t)x^*(t-\tau)\,dt$ $\tau=0$

  $\int_{-\infty}^{\infty} x(t)x^*(t)\,dt = E = \int_{-\infty}^{\infty} |x(t)|^2\,dt$

- If τ is increased in either directions, the autocorrelation $R(\tau)$ decreases. τ is increases $R(\tau)$ increases. $|R(\tau)|$ is maximum it $\tau=0$.

  Mathematically $|R(\tau)| \le R(0)\ \forall\ \tau$

- Auto correlation function and energy spectral densities are Fourier transform pairs. i.e.

$$\text{F.T}\,[R(\tau)]=\Psi(\omega)$$

$$\Psi(\omega)= \int_{-\infty}^{\infty} |R(\tau)\, e^{-j\omega\tau d\tau}\,dt$$

Autocorrelation Function of Power or periodic Signals:

- Consider a periodic signal x(t) with period T_0. The autocorrelation function of periodic power signal with period T is given by

  $R_{xx}(\tau)$ or $R(\tau)= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)x^*(t-\tau)\,dt$ if τ is negative direction

  $R(\tau)= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t+\tau)x^*(t)\,dt$

For a period, $R_{xx}(\tau)$ or $R(\tau)=\ \lim_{t\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}x(t)x^*(t-\tau)\,dt$

Properties of Autocorrelation Function of power Signal:

- Autocorrelation exhibits conjugate symmetry i.e. R (τ) = R*(-τ)

  Proof: Autocorrelation function of power signal is given by

  $R_{xx}(\tau)$ or $R(\tau)=\ \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}x(t)x^*(t-\tau)\,dt$

  $R^*(\tau)=\lim_{t\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}x(t-\tau)x^*(t)\,dt$

  $R^*(-\tau)=\lim_{t\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}x(t+\tau)x^*(t)\,dt\ = R(\tau)$

  R (τ) = R*(-τ)

- The value autocorrelation function of power signal is given at origin gives average power of that signal, which is given as:

  $R(0) = \frac{1}{T0}\int_{-\frac{T}{2}}^{\frac{T}{2}}|x(t)|^2\,dt$ =p

  Proof:

  $R_{xy}(\tau)$ or $R(\tau)=\ \frac{1}{T0}\int_{-\frac{T}{2}}^{\frac{T}{2}}x(t)x^*(t-\tau)\,dt$ τ=0

  $R(0)=\ \frac{1}{T0}\int_{-\frac{T}{2}}^{\frac{T}{2}}x(t)x^*(t)\,dt$

  $=\frac{1}{T0}\int_{-\frac{T}{2}}^{\frac{T}{2}}|x(t)|^2\,dt$ =p

- If τ is increased in either directions, the autocorrelation R(τ) decreases. τ is increases R(τ) increases. |R(τ)| is maximum it τ=0.
- Auto correlation function and power spectral densities are Fourier transform pairs. i.e.

  F.T [R(τ)]=S(ω)

  Auto correlation function is a periodic R (τ) = R(τ+nT_0),n=1,2,3,------

Cross Correlation Function:

- Cross correlation is the measure of similarity between two different signals.

  Consider two signals $x_1(t)$ and $x_2(t)$. The cross correlation of these two

Signals $R_{xy}(\tau)$ is given by

$$R_{xy}(\tau) \text{ or } R(\tau) = \int_{-\infty}^{\infty} x(t)y^*(t-\tau)\,dt$$

$$= \int_{-\infty}^{\infty} x(t+\tau)y^*(t)\,dt$$

$$\int_{-\infty}^{\infty} x(t)y^*(t)\,dt = 0$$

$$R_{xy}(0) = 0$$

Then the signals x(t) and y(t) are called orthogonal signals. Then the Cross correlation for

Orthogonal signals expressed as $R_{xy}(\tau)$ or $R(\tau) = \int_{-\infty}^{\infty} y(t)x^*(t-\tau)\,dt$

Properties of cross-correlation Function of Energy Signal:

- Cross-correlation exhibits conjugate symmetry i.e. $R_{xy}(\tau) = R_{xy}^*(\tau)$
  Cross-correlation is not a commutative $R_{xy}(\tau) \neq R_{xy}(\tau)$
- The value autocorrelation function of energy signal is given at origin i.e. at $\tau=0$ is equal to total energy of that signal, which is given as:
  $R_{xy}(0) = E = 0$ i.e $\int_{-\infty}^{\infty} x(t)y^*(t)\,dt=0$, Then the signals x(t) and y(t) are

called orthogonal Signals.

- The cross-correlation of energy signal corresponds to the multiplication of the FT of one signal and complex conjugate FT of another signal
  $R_{xy}(\tau)=X(j\omega)y^*(j\omega)$ called correlation theorem.

Properties of Cross-correlation Function of power Signal:

- Consider a periodic signal x(t) with period T_0.The cross- correlation function of periodic power signal with period T is given by

$$R_{xy}(\tau) \text{ or } R(\tau) = \frac{1}{T0}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)y^*(t-\tau)\,dt$$

$$R_{yx}(\tau) = \frac{1}{T0}\int_{-\frac{T}{2}}^{\frac{T}{2}} y(t)x^*(t-\tau)\,dt$$

- Cross-correlation exhibits conjugate symmetry i.e. $R_{xy}(\tau)) = R_{yx}(\tau)$

  Proof: Cross-correlation function between 2 signals x(t) and y(t)

$$R_{xy}(\tau) \text{ or } R(\tau) = \int_{-\infty}^{\infty} x(t)y^*(t-\tau)\,dt \quad \text{put} \quad p = t-\tau$$

$$R_{yx}(\tau) = \int_{-\infty}^{\infty} x(p+\tau)y^*(p)\,dt$$

$R_{xy}(\tau)$ or $R(\tau)= \int_{-\infty}^{\infty} y(t)x^*(t-\tau)\,dt$ substituting p= t

$R(\tau)= \int_{-\infty}^{\infty} y(p)x^*(p-\tau)\,dt$

$R^*_{xy}(\tau)$ or $R(\tau)= \int_{-\infty}^{\infty} x(p+\tau)y^*(p)\,dt$

$R_{xy}(\tau) = R^*_{yx}(\tau)$

- Cross-correlation is not a commutative $R_{xy}(\tau) \neq R_{xy}(\tau)$

  If $R_{xy}(0)=0$, $R_{xy}(\tau)$ or $R(\tau)= \frac{1}{T0}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)y^*(t)\,dt = 0$

  Then the signals x(t) and y(t) are called orthogonal Signals.

Relation between convolution and correlation:

- In correlation, physical time 't' plays the role of a dummy variable and it disappears after solving the integral. But in convolution , delay parameter, τ plays the roles of a dummy variable.
- Correlation $R_{xy}(\tau)$ is a function of the delay parameter τ, where as convolution is a function of time.
- Correlation can be obtained by convolving x(-t) and y*(t)

Relation between auto correlation function R (τ) and energy/power spectral density function ESD Ψ(f) and PSD s(f)

- auto correlation function R (τ) of an energy signal and its ESD, Ψ(f) forms a FT pair,

  $R(\tau) \longleftrightarrow \Psi(f)$

  Proof: the Cross-correlation function between 2 signals x(t) and y(t)is given as

  $R_{xy}(\tau)$ or $R(\tau)= \int_{-\infty}^{\infty} X(f)Y^*(f)e^{j\omega\tau}\,df$

  If both functions are same then the autocorrelation is given by

  $R_{xx}(\tau)$ or $R(\tau)= \int_{-\infty}^{\infty} X(f)x^*(f)e^{j\omega t}\,dt=\int_{-\infty}^{\infty} |x(f)|^2\, e^{j\omega\tau}df$

  $F^{-1}=[\,|F(f)|^2]$

  $|F(f)|^2 = \Psi(f)$

  $R(\tau)= F^{-1}\,[\Psi(f)]$, $F[R(\tau)]= [\Psi(f)]$

- auto correlation function R (τ) of an power signal and its PSD, S(f) forms a FT pair,

  R (τ) $\longleftrightarrow$ S(f)

Proof: the Cross-correlation function of a power signal x(t) in terms of FSC

$$R(\tau)= \sum_{-\infty}^{\infty} X_n X_{-n} e^{jn\omega 0\tau} \qquad X_n X_{-n} \text{ are FSC.}$$

$$R(\tau)= \sum_{-\infty}^{\infty} |X_n|^2 e^{jn\omega 0\tau}$$

Taking FT on both sides

$$F[R(\tau)]= \int_{-\infty}^{\infty} [\sum_{-\infty}^{\infty} |X_n|^2 e^{jn\omega 0\tau}] e^{-j\omega\tau} d\,\tau$$

Inter change summation and integration

$$F[R(\tau)] = \sum_{-\infty}^{\infty} |X_n|^2 \int_{-\infty}^{\infty} [\sum_{-\infty}^{\infty} e^{-j\tau(\omega - jn\omega 0)} d\,\tau = 2\pi \sum_{-\infty}^{\infty} |X_n|^2\ \delta(\omega - jn\omega 0)$$

$$= \sum_{-\infty}^{\infty} |X_n|^2\ \delta(f - jnf0)$$

The RHS is the PSD S(ω) or S(f) of periodic function x(t)

F[R (τ)]= S(f)

[R(τ)]= F^{-1}[S(f)]

So R (τ) $\longleftrightarrow$ S(f)

UNIT VI

THE LAPLACE TRANSFORM

Learning Objectives:

- To introduce the concept of various Laplace transform
- To introduce the concept of convergence of Laplace transform.
- To introduce the concept of representation of a Laplace transform in s domain

Learning Outcomes:

Students will be able to

- Apply properties of Laplace transform
- Solving the integro differential equations using Laplace transform
- Obtain the Laplace transform of standard signals

Syllabus:

Laplace transform of signals, Convergences of Laplace transform, Properties of region of convergence (ROC),Unilateral Laplace transform, Properties Unilateral Laplace transform, Initial value theorem, Final value theorem, Inversion of Unilateral and Bilateral Laplace transform, Relation between Laplace and Fourier Transforms

Advantages of Laplace transform:

1Signals which are not convergent in Fourier transform are convergent in Laplace transform.

2convolutions in time domain can be obtained by simple multiplication in S domain

3 Integro differential equations are converted into simple algebraic equations

Laplace transform of signals:

The Laplace transform of a function$f(t)$, defined for all real numbers $t \geq 0$, is the function $F(s)$, which is a unilateral transform defined by

$$L[x(t)] = X(s) = \int_0^{\infty} x(t) e^{-st} dt \qquad (s = \sigma + j\omega)$$

Bilateral Laplace Transform

The *Bilateral Laplace Transform* is defined as follows:

$$x(t) = X(S) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

Inverse Laplace Transform:

The inverse Laplace transform is defined as

$$x(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) e^{st} ds$$

Conditions for Existence of Laplace Transform:

Dirichlet's conditions are used to define the existence of Laplace transform. i.e.

- The function f(t) has finite number of maxima and minima.
- There must be finite number of discontinuities in the signal f(t),in the given interval of time.
- It must be absolutely integrable in the given interval of time. i.e. $\int_{-\infty}^{\infty}|f(t)|dt<\infty$

Relationship between fourier transorm and laplace transform:

$$\text{Laplace transform of } x(t) = X(S) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

Substitute s= σ + jω in above equation.

$$\rightarrow X(\sigma + j\omega) = \int_{-\infty}^{\infty} x(t)e^{-(\sigma+j\omega)t}dt$$

$$= \int_{-\infty}^{\infty} [x(t)e^{-\sigma t}]e^{-j\omega t}dt$$

$$\therefore X(S) = F.T[x(t)e^{-\sigma t}] \ldots \ldots (2)$$

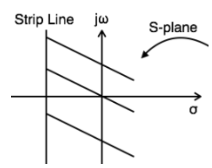
$$X(S) = X(\omega) \qquad for\ s = j\omega$$

i.e., Fouriertransform is a special case of Laplace transform when Re[s] or σ =0

Region of convergence:

The range variation of σ for which the Laplace transform converges is called region of convergence.

Properties of ROC of Laplace Transform:



- ROC contains strip lines parallel to jω axis in s-plane.
- If x(t) is absolutely integral and it is of finite duration, then ROC is entire s-plane.
- If x(t) is a right sided sequence then ROC : Re{s} $>\sigma_o$.
- If x(t) is a left sided sequence then ROC : Re{s} $<\sigma_o$.
- If x(t) is a two sided sequence then ROC is the combination of two regions.

Comparison between Fourier transform and Laplace transform

Sl.No	**Fourier Transform**	**Laplace Transform**

1	The fourier transform of a function x(t) can be represented by a continuous sum of exponential functions of the form $e^{j\omega t}$, i.e. it uses summation of waves of +ve and –ve frequencies.	The Laplace transform of a function x(t) can be represented by a continuous sum of complex exponential damped waves of the form e^{st}.
2	The fourier transform technique is applied for solving differential equations that relate the input and output.	The Laplace transform technique is also applied for solving differential equations that relate the input and output.
3	It is rarely used for solving differential equations because fourier transform does not exist for many signals as $\lvert x(t) \rvert$ is not absolutely integrable.	It is very widely used for solving differential equations because Laplace transform exists even for signals for which the fourier transform does not exist.
4	It does not have any convergence factor.	It has a convergence factor and so is more general.
5	Fourier transform cannot be used to analyze unstable systems.	Laplace transform can be used to analyze even unstable systems
6	Fourier transform is the Laplace transform evaluated along the imaginary axis of the s-plane, i.e. $X(\omega)= X(s)\rvert_{s=j\omega}$.	Laplace transform of x(t) is the fourier transform of $x(t)e^{-\sigma t}$, i.e. $L[x(t)]= F[x(t)e^{-\sigma t}]$

PROPERTIES OF LAPLACETRANSFORM:

Property	Definition
Linearity	$\mathcal{L}\{af(t)+bg(t)\} = aF(s)+bG(s)$
Differentiation	$\mathcal{L}\{f'\} = s\mathcal{L}\{f\} - f(0^-)$ $\mathcal{L}\{f''\} = s^2\mathcal{L}\{f\} - sf(0^-) - f'(0^-)$ $\mathcal{L}\left\{f^{(n)}\right\} = s^n\mathcal{L}\{f\} - s^{n-1}f(0^-) - \cdots - f^{(n-1)}(0^-)$
Frequency Division	$\mathcal{L}\{tf(t)\} = -F'(s)$ $\mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s)$
Frequency Integration	$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(\sigma)\,d\sigma$
Time Integration	$\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \mathcal{L}\{u(t)*f(t)\} = \frac{1}{s}F(s)$
Scaling	$\mathcal{L}\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right)$
Initial value theorem	$f(0^+) = \lim_{s\to\infty} sF(s)$
Final value theorem	$f(\infty) = \lim_{s\to 0} sF(s)$
Frequency Shifts	$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$ $\mathcal{L}^{-1}\{F(s-a)\} = e^{at}f(t)$
Time Shifts	$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s)$ $\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)u(t-a)$
Convolution Theorem	$\mathcal{L}\{f(t)*g(t)\} = F(s)G(s)$

INVERSE LAPLACE TRANSFORM:

The transfer function may of the form

$$\frac{b_0 s^m + b_1 s^{m-1} + \ldots + b_{m-1}s + b_m}{s^n + a_1 s^{n-1} + \ldots a_{n-1}s + a_n} e^{-\tau s}$$

Consider the proper partial fractions for the above expression * Summary of Partial–Fraction Expansion

(1) Expansion Structure:

Simple Roots (including complex conjugate)

=> $\frac{A_j}{s-\alpha_j}$ could be complex.

Repeated Roots: m multiplicity

=> $\frac{B_1}{s-\beta_j}+\frac{B_2}{(s-\beta_j)^2}+\ldots+\frac{B_m}{(s-\beta_j)^m}$

(2) Avoid complex number

For complex conjugates: $\alpha_j = a + jb$

$$\frac{A_j}{s-\alpha_j}+\frac{A_j^*}{s-\alpha_j^*} \Rightarrow \frac{cs+D}{(s-a)^2+b^2}$$

$$\frac{B_k}{(s-\alpha_j)^k}+\frac{B_k^*}{\left(s-\alpha_j^*\right)^k} \Rightarrow \frac{cs+D}{[(s-a)^2+b^2]^k}$$

LAPLACE TRANSFORM OF STANDARD FUNCTIONS:

Entry	**Laplace Domain**	**Time Domain** (note)
unit impulse	1	$\delta(t)$ unit impulse
unit step	$\Gamma(s)=\frac{1}{s}$	$\gamma(t)$ (note)
ramp	$\frac{1}{s^2}$	t
parabola	$\frac{2}{s^3}$	t^2
t^n (n is integer)	$\frac{n!}{s^{(n+1)}}$	t^n
exponential	$\frac{1}{s+a}$	e^{-at}
time multiplied exponential	$\frac{1}{(s+a)^2}$	te^{-at}
Asymptotic exponential	$\frac{1}{s(s+a)}$	$\frac{1}{a}\left(1-e^{-at}\right)$

double exponential	$\frac{1}{(s+a)(s+b)}$	$\frac{e^{-at}-e^{-bt}}{(b-a)}$
asymptotic double exponential	$\frac{1}{s(s+a)(s+b)}$	$\frac{1}{ab}\left(1-\frac{be^{-at}-ae^{-bt}}{(b-a)}\right)$
asymptotic critically damped	$\frac{1}{s(s+a)^2}$	$\frac{1}{a^2}\left(1-e^{-at}-ate^{-at}\right)$
sine	$\frac{\omega_0}{s^2+\omega_0^2}$	$\sin(\omega_0 t)$
cosine	$\frac{s}{s^2+\omega_0^2}$	$\cos(\omega_0 t)$
decaying sine	$\frac{\omega_d}{(s+a)^2+\omega_d^2}$	$e^{-at}\sin(\omega_d t)$
decaying cosine	$\frac{s+a}{(s+a)^2+\omega_d^2}$	$e^{-at}\cos(\omega_d t)$